

An Application of the Browder-Minty Theorem in a Problem of Partial Differential Equations

Uma Aplicação do Teorema de Browder-Minty a um Problema de Equações Diferenciais Parciais

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Abstract

In this work, we will show existence of weak solution for a semilinear elliptic problem using as main tool the Browder-Minty Theorem. First, we will make a brief introduction about basic theory of the Sobolev Spaces to support our study and provide sufficient tools for the development of our work. Then we will take a quick approach on the Browder-Minty Theorem and use this result to show the existence of at least one weak solution to an elliptic Partial Differential Equations (PDE) problem whose nonlinearity, denoted by f , is a known function. For this, in addition to the already mentioned results, we will also use as study tools: Embedding Sobolev Theorems, Linear Continuous Operators Theory, Poincaré Inequality and Hölder Inequality.



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Resumo

Neste trabalho, mostraremos a existência de uma solução fraca para um problema elíptico semilinear utilizando como ferramenta principal o Teorema de Browder-Minty. Primeiro, faremos uma breve introdução sobre a teoria básica dos Espaços de Sobolev com o objetivo de fundamentar nosso estudo e fornecer ferramentas suficientes para o desenvolvimento do nosso trabalho. Em seguida, faremos uma abordagem rápida sobre o Teorema de Browder-Minty e utilizaremos esse resultado para mostrar a existência de pelo menos uma solução fraca para um problema de Equações Diferenciais Parciais (EDP) elípticas cuja não-linearidade, denotada por f , é uma função conhecida. Para isso, além do resultado já mencionado, também utilizaremos como ferramentas de estudo: Teoremas de Imersão de Sobolev, Teoria dos Operadores Lineares e Contínuos, Desigualdade de Poincaré e Desigualdade de Hölder.

1 Introduction

In this paper, we will study existence of solution to the following problem of Partial Differential Equations:

$$-\Delta u = \lambda f(u) \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega, \quad (1)$$

where Ω is a bounded regular domain¹ of \mathbb{R}^N ($N > 3$) with boundary $\partial\Omega$, λ is a positive real parameter, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a known function and Δ is the Laplacian operator

$$\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}, \quad x_i \in \Omega. \quad (2)$$

The study of Partial Differential Equations is a subject of great importance in Mathematics due to its great applicability in the most diverse areas. An infinity of problems resulting from the study of the Physics, Chemistry, Biology and Engineering can be modeled in terms of Partial Differential Equations submitted to some boundary condition. An important type of these partial equations are nonlinear problems involving elliptic operators. For these problems, we can find several solubility techniques. In a special way, we can guarantee the existence of solutions to these problems by topological methods (Degree Theory, Fixed-Point Theory and Browder-Minty Theorem, for example) and variational methods (Methods of Sub-Super Solution, Minimization Arguments, Mountain Pass Theorem, and others).

¹A **domain** Ω is a connected open subset in \mathbb{R}^N . For definition of **regular domain** see Adams (1975, p. 84) or Figueiredo (1977, p. 4). A domain Ω is said to be **bounded** if exists a constant $R > 0$ such that $\Omega \subseteq B_R$, where B_R be the ball of radius R .

More precisely, this work concerns existence of weak solution² to the semilinear elliptic problem:

$$-\Delta u = \lambda \sin u \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega, \quad (3)$$

where $\Omega \subset \mathbb{R}^N$ ($N > 3$) is a bounded regular domain with boundary $\partial\Omega$ and $\lambda \in \mathbb{R}$ be a positive parameter. In this case, we will determine conditions on λ so that problem (3) admits weak solution. For this, we will use as main techniques the Browder-Minty Theorem and Sobolev Spaces Theory.

Semilinear elliptic problems in the form (1) are often studied in Partial Differential Equations on various forms that is on various conditions about the function f and different boundary conditions (Dirichlet, Neumann, Steklov and mixed conditions). In the particular case where $f(t) = \sin t$, Castro (1980), using Index Theory results and Deformation Lemma, showed existence of multiple weak solutions to the problem (3) when $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of the linear problem

$$-\Delta u = \lambda u \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega. \quad (4)$$

In 2011, Badiale and Serra studied existence of weak solutions to problem (1) in case where f is a bounded function. More precisely, using Global Minimization techniques for coercive problems, the authors have shown that there is at least one weak solution to problem (1) when f is a bounded function.

Motivated by these results, we will show the existence of at least one weak solution to the problem (3) obtained via Browder-Minty Theorem. The Sobolev Spaces results and the Browder-Minty Theorem will be presented in the next section. In addition, we will indicate basic bibliographies where readers can find a complete proof of the version of Browder-Minty Theorem that we will use in this work. We highlight that the results of this work are the result of a research of a project of Scientific Initiation (PIBIC) that is still under development in the Instituto Federal Goiano - *Campus* Urutaí in 2019. Thus, the results presented here are a part of the research that is still being developed and, with this, a more improved result may arise after the project completion.

2 Theoretical References

In this section, we will present a little introduction about the basic theory that we will use as main tool for obtaining the results. The theory presented here is based on the ideas of Brézis (2011) for Sobolev Spaces, Figueiredo (1977) and Deimling (1985) for monotone operators.

²See Definition 2.3.

2.1 Sobolev Spaces

Let Ω be a bounded regular domain in \mathbb{R}^N with boundary denoted by $\partial\Omega$ and consider the function spaces $L^2(\Omega)$ and $H^1(\Omega)$ given by

$$L^2(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable; } \int_{\Omega} u^2 dx < +\infty \right\} \quad (5)$$

and

$$H^1(\Omega) = \left\{ u \in L^2(\Omega); \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, \dots, N \right\}. \quad (6)$$

The space $L^2(\Omega)$ is called Lebesgue Space. The inner product and norm in $L^2(\Omega)$ are respectively given by

$$\langle u, v \rangle_2 = \int_{\Omega} uv dx \quad \text{and} \quad \|u\|_2 = \left(\int_{\Omega} u^2 dx \right)^{\frac{1}{2}}, \quad u, v \in L^2(\Omega). \quad (7)$$

On the other hand, the space $H^1(\Omega)$ is called Sobolev Space with inner product and norm given, respectively, by

$$\langle u, v \rangle_H = \langle u, v \rangle_2 + \sum_{i=1}^N \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle_2 \quad \text{and} \quad \|u\|_H = \left(\|u\|_2^2 + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right)^{\frac{1}{2}}, \quad u, v \in H^1(\Omega). \quad (8)$$

The space $H^1(\Omega)$ equipped with inner product $\langle \cdot, \cdot \rangle_H$ and standard norm $\|\cdot\|_H$ is a Hilbert Space³. Now, consider the subspace of $H^1(\Omega)$, denoted by $H_0^1(\Omega)$, defined by

$$H_0^1(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } \partial\Omega\}. \quad (9)$$

More precisely, $H_0^1(\Omega)$ is the closure of C^∞ functions space with compact support⁴ in Ω , in relation to norm $\|\cdot\|_H$ of the space $H^1(\Omega)$. $H_0^1(\Omega)$ is natural space where we will look by weak solutions of the problem (3). The following theorem deals with some important inequalities that we will use frequently throughout the proof of the main result. For proof of this result, the reader is advised to refer to Brézis (2011) and Castro (1980).

Theorem 2.1. *Let Ω a bounded domain in \mathbb{R}^N .*

(i) **(Hölder Inequality)** *If $u, v \in L^2(\Omega)$, then:*

$$\int_{\Omega} uv dx \leq \left(\int_{\Omega} u^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} v^2 dx \right)^{\frac{1}{2}}. \quad (10)$$

³ H is a Hilbert Space if H is a vectorial space provided with inner product and complete with the norm $\|\cdot\|_H$.

⁴The support of a function u , denoted by $\text{supp}(u)$, is the closure in Ω of the set $\{x \in \Omega; u(x) \neq 0\}$.

(ii) **(Poincaré Inequality)** If $u \in H_0^1(\Omega)$, then there is a constant $c > 0$ such that

$$\int_{\Omega} |u|^2 dx \leq c \int_{\Omega} |\nabla u|^2 dx. \quad (11)$$

As usualy, the space $H_0^1(\Omega)$ will be equipped with inner product and norm given by

$$\langle u, v \rangle = \int_{\Omega} \nabla u \nabla v dx, \quad u, v \in H_0^1(\Omega) \quad \text{and} \quad \|u\| = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}, \quad u \in H_0^1(\Omega). \quad (12)$$

The norm $\|\cdot\|$ given in equation (12) is equivalent to norm $\|\cdot\|_H$ of the space $H^1(\Omega)$ being motivated by Poincaré Inequality. It's important remember, from Sobolev Spaces Theory, that $(L^2(\Omega), \|\cdot\|_2)$ and $(H_0^1(\Omega), \|\cdot\|)$ are Hilbert Spaces and consequently reflexives⁵. In addition, by Rellich-Kondrachov Embedding Theorem (MEDEIROS; MIRANDA, 2000, p. 79, Theorem 2.5.4), the application $i : H_0^1(\Omega) \rightarrow L^2(\Omega)$ is compact and there is a constant $c > 0$ such that

$$\|u\|_2 \leq c \|u\|, \quad u \in H_0^1(\Omega). \quad (13)$$

Indeed, by Laplacian Operator Spectral Theory, it can be shown that the constant c given in inequality (13) is inverse of first eigenvalue λ_1 of linear problem (4), since λ_1 is defined by

$$\lambda_1 = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|u\|^2}{\|u\|_2^2}. \quad (14)$$

The following theorem deals about convergence of a sequence in Lebesgue Spaces. For a proof of this result see Brézis (2011, p. 90).

Theorem 2.2 (Dominated Convergence Theorem, Lebesgue). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $(u_n)_{n \in \mathbb{N}}$ be a sequence of functions in $L^1(\Omega)$ such that:

- (i) $u_n(x) \rightarrow u(x)$ almost everyone (a.e.) on Ω ;
- (ii) there is a function $h \in L^1(\Omega)$ such that $|u_n(x)| \leq h(x)$ a.e. on Ω .

Then, $u \in L^1(\Omega)$ and $u_n \rightarrow u$ in $L^1(\Omega)$.

For more details on Sobolev Spaces and Laplacian Operator Spectral Theory see Adams (1975), Brézis (2011), Figueiredo (1977) and Lório Jr., Lório (2010). We shall now define what is meant by weak solution for elliptic problems of the form (1).

A function $u_0 \in H_0^1(\Omega)$ is a weak solution of the problem (1) if u_0 satisfies the following equality:

$$\int_{\Omega} \nabla u_0 \nabla v dx = \lambda \int_{\Omega} f(u_0) v dx, \quad (15)$$

⁵A Banach Spaces E is called reflexive if and only if every bounded sequence in E admits a weakly convergent subsequence.

for all function $v \in H_0^1(\Omega)$ (BADIALE; SERRA, 2011).

Thus, we define:

Definition 2.3. A function $u_0 \in H_0^1(\Omega)$ is a weak solution of the problem (3) if u_0 satisfies:

$$\int_{\Omega} \nabla u_0 \nabla v dx = \lambda \int_{\Omega} (\sin u_0) v dx, \quad v \in H_0^1(\Omega). \quad (16)$$

Let $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the functional energy associated with problem (3) given by

$$J(u) = \frac{1}{2} \|u\|^2 - \lambda \int_{\Omega} F(u) dx, \quad u \in H_0^1(\Omega), \quad (17)$$

where

$$F(u) = \int_0^u \sin(t) dt = -\cos u + 1 \quad (18)$$

is the potential of function $f(u) = \sin u$. Therefore, J is a C^1 -functional (BRÉZIS, 2011) and has derivate $J' : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ given by

$$J'(u)v = \langle u, v \rangle - \lambda \int_{\Omega} (\sin u) v dx, \quad u, v \in H_0^1(\Omega), \quad (19)$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ defined by

$$H^{-1}(\Omega) = \{f : H_0^1(\Omega) \rightarrow \mathbb{R}; f \text{ is linear and continuous}\}.$$

Note that critical points of functional J are weak solutions of the problem (3). Indeed, if u_0 is a critical point of J then $J'(u_0)v = 0$ for all $v \in H_0^1(\Omega)$. From this and by definition of J' follows that equation (16) is satisfied. Thus u_0 is a weak solution of the problem (3). Therefore, we will search for critical points of the functional J .

In the next section, will be studied the Browder-Minty Theorem.

2.2 Monotone Operators and Browder-Minty Theorem

In this section, we will address some concepts related to the study of monotone, coercive, bounded operators and about Browder-Minty Theorem. We will start with basic definitions of operators in Banach Spaces.

Definition 2.4 (FIGUEIREDO (1977)). Let E be a Reflexive and Real Banach Space⁶, E^* the dual of E and $F : E \rightarrow E^*$ be a operator.

⁶ E is a Banach Space if E is a Complete and Normad Vetorial Space.

(i) (Monotonicity) F is **monotone** if:

$$\langle F(u) - F(v), u - v \rangle \geq 0, \quad u, v \in E. \quad (20)$$

(ii) (Continuity) F is **continuous** if for every sequence (u_n) in E such that $u_n \rightarrow u_0$ in E we have $F(u_n) \rightarrow F(u_0)$ in E^* . In particular, F is **hemicontinuous** if $F(u + t_n v) \rightarrow F(u)$ when $t_n \rightarrow 0$ for all $u, v \in E$.

(iii) (Coercivity) F is **coercive** if:

$$\lim_{\|u\| \rightarrow \infty} \frac{F(u)u}{\|u\|} = +\infty, \quad u \in E. \quad (21)$$

The following theorem guarantees the existence of a solution to equations of the form $Fu = b$, where $b \in E^*$ and F satisfies monotonicity, continuity and coercivity conditions. This result is known as **Browder-Minty Theorem**.

Theorem 2.5 (Browder-Minty). *Let E be a Reflexive and Real Banach Space, E^* the dual of E and $F : E \rightarrow E^*$ be a monotone, hemicontinuous and coercive operator. Then, F is surjective⁷ in E^* .*

According to Figueiredo (1977, p. 133), the Browder-Minty Theorem has a simple geometric sense in case $E = \mathbb{R}$; it simply says that if a increasing continuous monotone function $f : \mathbb{R} \rightarrow \mathbb{R}$ is such that $f(x) \rightarrow \pm\infty$ according to $x \rightarrow \pm\infty$, then f is surjective. In this case, this is a consequence of the Intermediate Value Theorem.

For a complete proof of the Browder-Minty Theorem see Figueiredo (1977, p. 132), Deimling (1985, p. 117) and Carl, Le, Montreanu (2005, p. 41).

Applications of the Browder-Minty Theorem are found in the search for solutions of problems in the form $\mathcal{A}u = f$, where \mathcal{A} is a continuous, coercive and monotone operator and f satisfies certain growth conditions. In the case where \mathcal{A} takes the form of p-Laplacian operator⁸, for example, the Browder-Minty Theorem is used to show that $\mathcal{A} : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ ⁹ defined by

$$\langle \mathcal{A}u, v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx, \quad u, v \in W_0^{1,p}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 1, \quad (22)$$

is a homeomorphism and, consequently, ensure existence of a solution to equation

$$\langle \mathcal{A}u, v \rangle = \int_{\Omega} f v dx, \quad v \in W_0^{1,p}(\Omega). \quad (23)$$

⁷A operator $F : E \rightarrow E^*$ is surjective if for each $b \in E^*$ there exists $u \in E$ such that $Fu = b$.

⁸The p-Laplacian operator is defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < N$.

⁹ $W_0^{1,p}(\Omega)$ is the Sobolev Space with dual $W^{-1,p'}(\Omega)$ (see Brézis (2011, p. 263)).

Other elliptic problems solved by using the Browder-Minty Theorem can be found in Le and Schmitt (1998). In this work, the authors use the Browder-Minty Theorem and Sub-Super solution Methods to show existence of solution of a class of degenerate quasilinear elliptic problems. For other details on the Browder-Minty Theorem and applications see Chipot (2012) and Figueiredo (1977).

3 Methodology

The methodology used for the development of this paper was based on bibliographical research in mathematical literature on the subject related in “References” section. Basically, this work was motivated by classical studies carried out in Partial Differential Equations. In general, to show existence of weak solution to the problem (3) we chose to use the Browder-Minty Theorem as main tool since such a result isn't used frequently in literature to solve this class of problems. Usually, due the limitation of the sine function, Minimization Techniques or Min-Max Theorems are chosen such as the Rabinowitz Mountain Pass Theorem. The use of this result as technique is our main contribution to the study of these semilinear problems.

First, we will associate with the problem (3) a functional energy, denoted by J , whose critical points are related with the weak solutions of (3) as follows: if u_0 is a critical point of functional J then u_0 is a weak solution of (3). Then, we will show that the derivate J' of J satisfies the monotonicity, continuity and coercivity conditions. Thus, we can apply the Browder-Minty Theorem to show that J' is surjective and the integral equation $J'(u)v = 0$ admits solution for any $v \in H_0^1(\Omega)$.

In addition to the Browder-Minty Theorem, we will use as tools for the development of our study the following classic results related to Functional Analysis: Sobolev Embedding Theorem (Rellich-Kondrachov Theorem), Lebesgue Dominated Convergence Theorem, Hölder Inequality, Poincaré Inequality and others. For completing, we will discuss other problems that may motivate and converge to new work related to the subject that we consider in this paper.

4 Results and Discussions

In this section, we will use the Browder-Minty Theorem to show existence of at least one solution to the problem (3). For this, let J the functional energy associated with the problem (3) given by equation (17) and with derivate given by

$$J'(u)v = \langle u, v \rangle - \lambda \int_{\Omega} (\sin u)v dx, \quad u, v \in H_0^1(\Omega). \quad (24)$$

Theorem 4.1. *Let $\lambda < \lambda_1$, then the problem (3) admits at least one weak solution.*

Proof: We will prove that J' is a monotone, continuous and coercive operator in $H^{-1}(\Omega)$ (dual of $H_0^1(\Omega)$). First, from the study of trigonometric functions, it's easy prove that,

$$\sin t - \sin s = 2 \sin \left(\frac{t-s}{2} \right) \cos \left(\frac{t+s}{2} \right), \quad t, s \in \mathbb{R}. \quad (25)$$

Therefore,

$$|\sin t - \sin s| = 2 \left| \sin \left(\frac{t-s}{2} \right) \right| \left| \cos \left(\frac{t+s}{2} \right) \right| \leq 2 \left| \frac{t-s}{2} \right| = |t-s|, \quad t, s \in \mathbb{R}, \quad (26)$$

and from this follows that $f(t) = \sin(t)$ is a contractive function, that is, f is a Lipschitz¹⁰ function with constant 1. We claim that J' is monotone. Indeed, using the definition of J' given in (24), the equality in (25) and Poincaré Inequality we have that

$$\begin{aligned} \langle J'(u) - J'(v), u - v \rangle &= \langle u - v, u - v \rangle - \lambda \int_{\Omega} (\sin u - \sin v)(u - v) dx \\ &\geq \|u - v\|^2 - \lambda \int_{\Omega} |\sin u - \sin v| |u - v| dx \\ &= \|u - v\|^2 - \lambda \|u - v\|_2^2 \\ &\geq \left(1 - \frac{\lambda}{\lambda_1}\right) \|u - v\|^2. \end{aligned} \quad (27)$$

As $\lambda < \lambda_1$ follows $\langle J'(u) - J'(v), u - v \rangle \geq 0$, for all $u, v \in H_0^1(\Omega)$, and therefore J' is monotone.

Now, note that J' is continuous. In fact, let (u_n) a sequence in $H_0^1(\Omega)$ such that $u_n \rightarrow u_0$ with $u_0 \in H_0^1(\Omega)$. We will prove $J'(u_n) \rightarrow J'(u_0)$ in $H^{-1}(\Omega)$. We know that

$$\|J'(u_n) - J'(u_0)\|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\| \leq 1}} |\langle J'(u_n) - J'(u_0), v \rangle|. \quad (28)$$

Using (26), Sobolev Embedding Theorem, Hölder Inequality and Poincaré Inequality we get

$$|\langle J'(u_n) - J'(u_0), v \rangle| \leq c_0 \|u_n - u_0\|_2 \|v\| + \lambda c_1 \|u_n - u_0\|_2 \|v\|, \quad (29)$$

where c_0 and c_1 are positive constants. Therefore

$$\|J'(u_n) - J'(u_0)\|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\| \leq 1}} |\langle J'(u_n) - J'(u_0), v \rangle| \leq c_0 \|u_n - u_0\|_2 + \lambda c_1 \|u_n - u_0\|_2. \quad (30)$$

¹⁰A function f is called Lipschitz if $|f(t) - f(s)| \leq c |t - s|$ for all t, s and for some constant $c > 0$.

As $u_n \rightarrow u_0$ in $H_0^1(\Omega)$ and $i : H_0^1(\Omega) \rightarrow L^2(\Omega)$ is a compact function, follows $i(u_n) \rightarrow i(u_0)$ and thus $u_n \rightarrow u_0$ em $L^2(\Omega)$, that is, $\|u_n - u_0\|_2 \rightarrow 0$. By (30),

$$\|J'(u_n) - J'(u_0)\|_{H^{-1}(\Omega)} \rightarrow 0, \quad (31)$$

and consequently $J'(u_n) \rightarrow J'(u_0)$ in $H^{-1}(\Omega)$. Therefore, J' is continuous.

In addition, J' is coercive. Indeed, by definition of J' we have that

$$J'(u)u = \|u\|^2 - \lambda \int_{\Omega} (\sin u)u dx \geq \|u\|^2 - \lambda \int_{\Omega} |u| dx \geq \|u\|^2 - \lambda c_2 \|u\|, \quad (32)$$

where c_2 is positive constant arising from the Hölder and Poincaré Inequalities. Of this,

$$\lim_{\|u\| \rightarrow \infty} \frac{J'(u)u}{\|u\|} \geq \lim_{\|u\| \rightarrow \infty} (\|u\| - \lambda c_2) = +\infty, \quad (33)$$

and therefore J' is coercive.

In short, we have proved that J' is a monotone, continuous and coercive operator. Applying Browder-Minty Theorem (Theorem 2.5) we have that J' is surjective and there exists $u_0 \in H_0^1(\Omega)$ such that $J'(u_0)v = 0$ for all $v \in H_0^1(\Omega)$. That is

$$\langle u_0, v \rangle = \lambda \int_{\Omega} (\sin u_0)v dx, \quad v \in H_0^1(\Omega). \quad (34)$$

Therefore, $u_0 \in H_0^1(\Omega)$ is a weak solution of the problem (3) and this prove the Theorem 4.1.

□

In addition, since the function $f(t) = \sin(t)$ is bounded, using the Schauder Regularity Theory (see Brézis (2011) and Gilbarg, Trudinger (1983)) we have that $u_0 \in C^{0,\alpha}(\bar{\Omega})$, for $\alpha \in (0, 1]$, where u_0 is the weak solution of the problem (3) obtained by Theorem 4.1. Therefore, u_0 is classical solution to this problem. It should be noted that the Browder-Minty Theorem does not guarantee uniqueness or multiplicity of solutions of the problem. For an analysis of these cases we need to search for additional tools in Nonlinear Analysis Theory.

A brief discussion about more general results: The methodology used to solve the problem (3) can also be used to solve more general results than the ones we detail with in this work.

(i) The semilinear elliptic problem

$$-\Delta_p u = \sin u \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega, \quad (35)$$

where Ω is a bounded domain in \mathbb{R}^N and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ($1 < p < N$), can be solved via Browder-Minty Theorem with similar techniques to those we have discussed here. The operator Δ_p is called p-Laplacian Operator and often appears in mechanics of fluids.

(ii) More generally, if f is a Lipschitzian function, that is, $|f(t) - f(s)| \leq c |t - s|$, $t, s \in \mathbb{R}$, with constant $c < \lambda_1^{-1}$, then the problem (1) admits weak solution for $\lambda < \lambda_1$ characterized as being a nontrivial minimum point of functional J .

5 Final Considerations

The search for solutions of semilinear elliptic problems in the form (1) is a comprehensive subject that has been discussed and studied for a long time in Partial Differential Equations. The decision of which method to use to find this solutions isn't immediate and depends on the characteristics of the problem (main operator and boundary conditions) and on the behavior of the functional energy associated, when the problem admits such an associated functional¹¹. Classical methods such as global (or local) minimization, Mountain Pass Theorem, Linking Theorem and Brouwer Degree are often used by various authors in mathematical literature of the subject. What we propose and develop in this work was the use of the Browder-Minty Theorem as tool to obtain a weak solution to the problem (3), since such results are not used as frequently as the mentioned above, due to the characteristics that the functional energy or its derivative must satisfy. In this sense, given the characteristics of the problem, we observed that Browder-Minty Theorem becomes an efficient tool to obtain a weak solution to our problem.

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¹¹For information on the existence of functional energy associated with elliptic problems see Adams (1975) and Brézis (2011).

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