Geometric approaches to the properties of the definite integral: enhancing understanding through visual demonstrations

Abordagens geométricas das propriedades da integral definida: aprimorando a compreensão através de demonstrações visuais

Enfoques geométricos en la demostracion de dos propiedades relativas a la integral definida: explorando la naturaleza intuitiva en las demostraciones

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Abstract: From a purely geometric perspective, this paper provides demonstrations of two properties related to the definite integral. The first one discusses the theorem of the definite integral of a function and its inverse. The second addresses the calculation of the area of a region rotated around the line $y = x$. The intention is to illustrate that when feasible, it is advantageous to explore the geometric aspects of the elements involved in a property for the purpose of its demonstration. In both cases, we assume only the monotonicity and continuity of the function.

Keywords: definite integral; inverse function; area of a symmetry region.

Resumo: A partir de uma perspectiva puramente geométrica, este artigo fornece demonstrações de duas propriedades relacionadas à integral definida. A primeira trata do teorema da integral definida de uma função e sua inversa. A segunda aborda o cálculo da área de uma região rotacionada em torno da reta $y = x$. A intenção é ilustrar que, quando viável, é vantajoso explorar os aspectos geométricos dos elementos envolvidos em uma propriedade para fins de sua demonstração. Em ambos os casos, assumimos apenas a monotonici-

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dade e a continuidade da função.

Palavras-chave: integral definida; função inversa; área de uma região de simetria.

Resumen: Desde una perspectiva puramente geométrica, este artículo proporciona demostraciones de dos propiedades relacionadas con la integral definida. La primera analiza el teorema de la integral definida de una función y su inversa. La segunda aborda el cálculo del área de una región rotada alrededor de la recta $y = x$. La intención es ilustrar que, cuando es factible, es ventajoso explorar los aspectos geométricos de los elementos involucrados en una propiedad con el propósito de demostrarla. En ambos casos, asumimos únicamente la monotonía y la continuidad de la función.

Palabras clave: integral definida; función inversa; área de una región de simetria.

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1 Introduction

Integral is a fundamental tool in calculus and mathematical analysis, playing a crucial role in a variety of fields, including physics, engineering, economics, and statistics, enabling the solution of problems related to the accumulation and variation of continuous quantities (Guidorizzi, 2015). Studying comprehensively is an interesting process, but it can be even more interesting, as many technological tools have been developed for this purpose.

It is needful for an educator to demonstrate ingenuity in the selection and effective utilization of appropriate instructional resources to enhance the mathematical aptitude of their students. As technology continues to advance, numerous options of instructional media for mathematics learning become available for this purpose (Sur, 2020).

Dynamic geometry and computer algebra systems have had a profound and transformative impact on the field of mathematics education (Hohenwarter; Fuchs, 2004), especially concerning the tools that aid in visualization.

In addition to choosing a tool that can assist in this process, it is necessary to delve into the geometric aspects of the elements involved in a property for the purpose of its demonstration. A powerful tool to assist with this is GeoGebra, as presented by Arini and Dewi (2019), and Narh-Kert and Sabtiwu (2022).

In this context, we aim to introduce an approach to illustrate properties involving definite integrals, using the geometric aspect, for a classic Theorem in the literature and provide an example of its application.

2 Geometric interpretation of the definite integral

Initially, we address the well-known relationship between the definite integral of a continuous function and the definite integral of its inverse function, which is given by Theorem 2.1, which can be found in Carlip (1991), Key (1994), Stewart, Clegg and Watson (2020). In Key (1994), demonstrations are presented of the relationship between the definite integral and its inverse, under the hypothesis that the function is monotone and with a continuous derivative. With the aim of showing the equivalence between the shell and disc methods in calculating the volume of a solid. Still in this same publication, the author adds that (Carlip, 1991) provided proof of this relationship assuming only the monotocity and continuity of the function. Exactly under these same conditions that we present the demonstration of this important result. Next we address the issue of calculating the area of a region delimited by a function whose graph is above the line $y = x$.

Theorem 2.1. *Let $f : [a, b] \to [c, d]$ be a function such that $f(a) = c$ and $f(b) = d$, continuous and differentiable with an inverse $f^{-1} : [c, d] \to [a, b]$, then*

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_c^d f^{-1}(y)dy. \tag{1}$$

Proof. *Assuming that $f^{-1}(y)$ is continuous, a proof of this Theorem can be performed by substituting $x = f^{-1}(y)$ and $dy = f'(x)dx$ into the integration by parts formula:*

$$\int f(x)dx = xf(x) - \int xf'(x)dx.$$

So that the corresponding definite integral becomes:

$$\int_a^b f(x)dx = xf(x) \mid_a^b - \int_a^b xf'(x)dx,$$

since

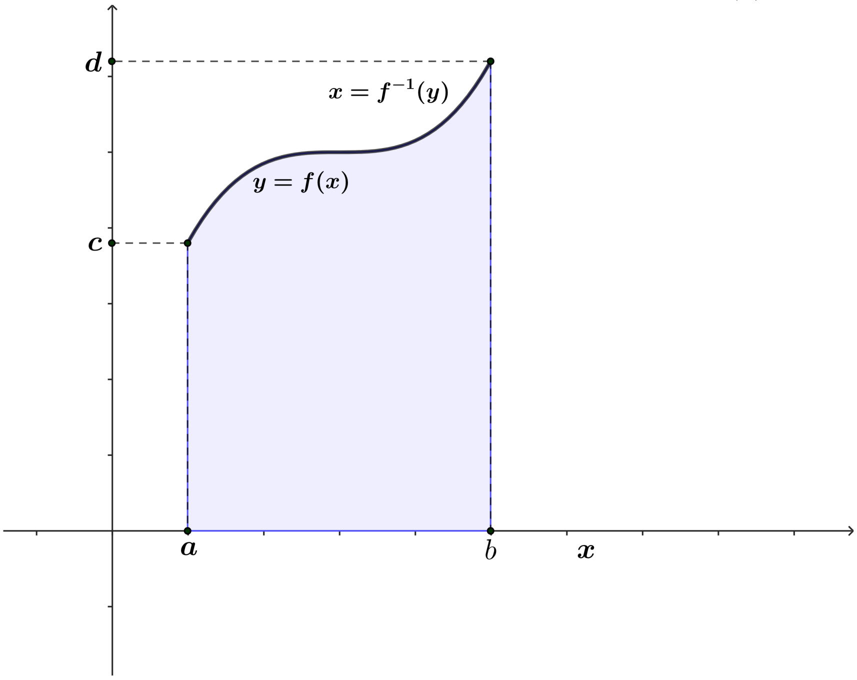
$$\int_a^b xf'(x)dx = \int_c^d f^{-1}(y)dy,$$

it follows that

$$\int_a^b f(x)dx = xf(x) \mid_a^b - \int_c^d f^{-1}(y)dy.$$

Therefore, we obtain result Eq. 1. This is a traditional approach commonly found in textbooks in general. However, we consider another type of proof for this Theorem 2.1. Initially, we address the well-known relationship between the definite integral of a continuous function and the definite integral of its inverse function preferably one that explores the geometric interpretation of the definite integral as well as the symmetry of the graphs of $f(x)$ and $f^{-1}(y)$ with respect to the identity line, as suggested by the following Figure 1.

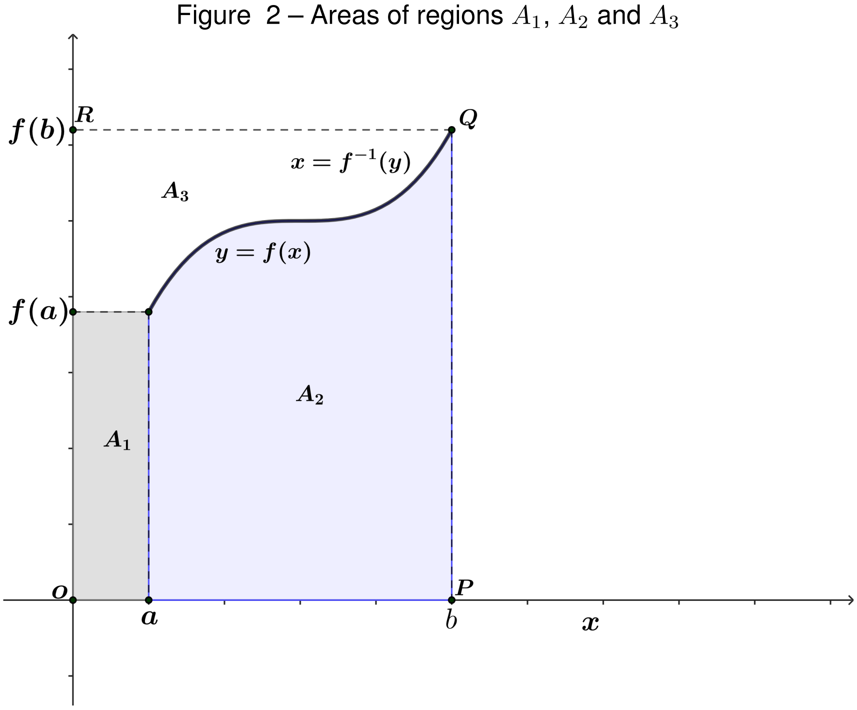
Figure 1 – Area of a region bounded by a curve $y = f(x)$



Source: Elaborated by the authors.

Proof. *From the Figure 2, we can see that the area of the rectangle OPQR, given by* $bf(b)$*, corresponds to the sum of the three shaded areas* $(A_1, A_2,$ *and* $A_3)$*, since* $A_1 = af(a)$ *and* $A_2 = \int_a^b f(x)dx$*.*

Figure 2 – Areas of regions A_1, A_2 and A_3



Source: Elaborated by the authors.

On the other hand, A_3 corresponds to the area of the region bounded by the curve $x = f^{-1}(y)$ and the lines $y = f(a)$ and $y = f(b)$, so

$$A_3 = \int_{f(a)}^{f(b)} f^{-1}(y)dy.$$

Then,

$$bf(b) = af(a) + \int_{f(a)}^{f(b)} f^{-1}(y)dy + \int_a^b f(x)dx.$$

Therefore,

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_c^d f^{-1}(y)dy.$$

This result opens up space for some very interesting questions, as exemplified in the following example.

Example 2.2. *Calculate*

$$\int_0^1 \left[\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3} \right] dx.$$

Naturally, judging by the structure of the integrating, it could not be evaluated using the various integration techniques covered in the Calculus I course. However, it is enough to observe that $y = f(x) = \sqrt[3]{1-x^7}$ defined into $[0, 1]$ is such that $f(0) = 1, f(1) = 0$, so $\sqrt[7]{1-y^3}$ it corresponds to its inverse, i.e. $\sqrt[7]{1-y^3} = x = f^{-1}(y)$. In this way, $f, f^{-1} : [0, 1] \to [0, 1]$, with $f(0) = 1$ and $f(1) = 0$. And thus, from Eq. (1), we have:

$$\int_0^1 f(x)dx = 1f(1) - 0f(0) - \int_1^0 f^{-1}(y)dy. \tag{2}$$

By the other side:

$$\begin{aligned} \int_0^1 \left[\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3} \right] dx &= \int_0^1 \sqrt[3]{1-x^7}dx - \int_0^1 \sqrt[7]{1-x^3}dx \\ &= \int_0^1 f(x)dx - \int_0^1 f^{-1}(y)dy \\ &= 1f(1) - 0f(0) \\ &= 0 \end{aligned}$$

Note that in the particular case of involution f of order 2, i.e., $f : [a, b] \to [a, b]$, such that $f^2(x) = f(f(x)) = x$, and thus are invertible, such that $f = f^{-1}$ (see Wiener and Watkins (1988)), it is verified that

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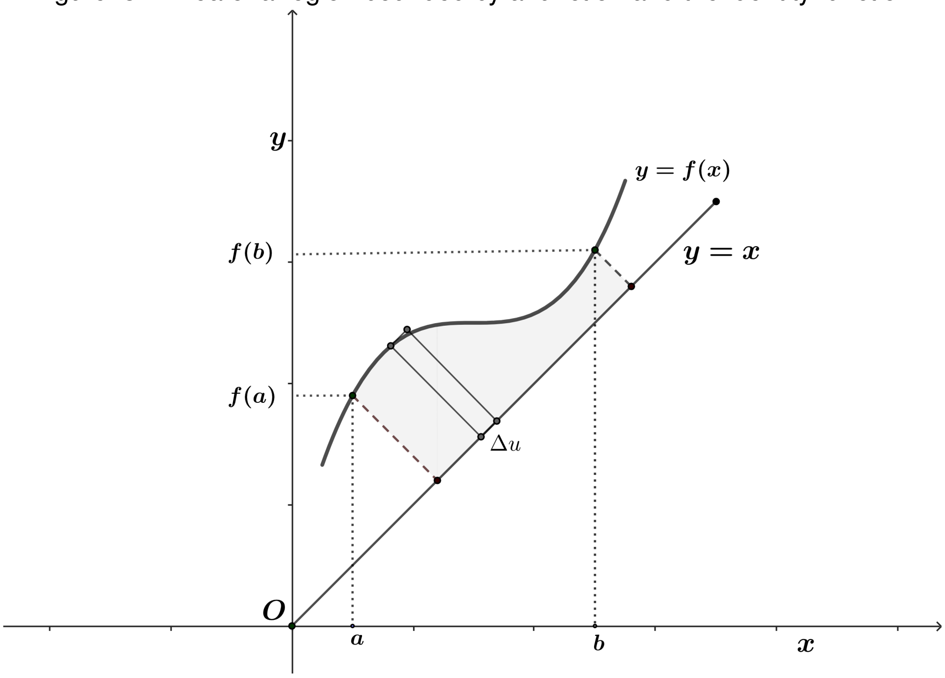
f is strictly decreasing (Lima, 2004). Therefore, $f(a) = b$ and $f(b) = a$. In this way, from Eq. 1, we have in particular

$$\int_a^b f(x)dx - \int_a^b f^{-1}(y)dy = 0.$$

Conventionally, it is common to calculate an area bounded by a function $y = f(x)$, the x-axes or y-axes, and the lines $x = a$ and $x = b$ or their corresponding problem in the case where $x = f(y)$. Now consider calculating an area bounded by a function f and an inclined line of the form $y = x$, i.e., a line that is not necessarily vertical or horizontal and is entirely below the curve represented by f, as suggested by the Figure 3.

Figure 3 – Area of a region bounded by a function and the identity function



Source: Elaborated by the authors.

The question and mathematical expression that allows calculating the region of this area was presented in Stewart, Clegg and Watson (2020), where a general version for any straight line is addressed. In this work, when considering the scenario where the straight line has a slope equal to 1 and passes through the origin, we obtain:

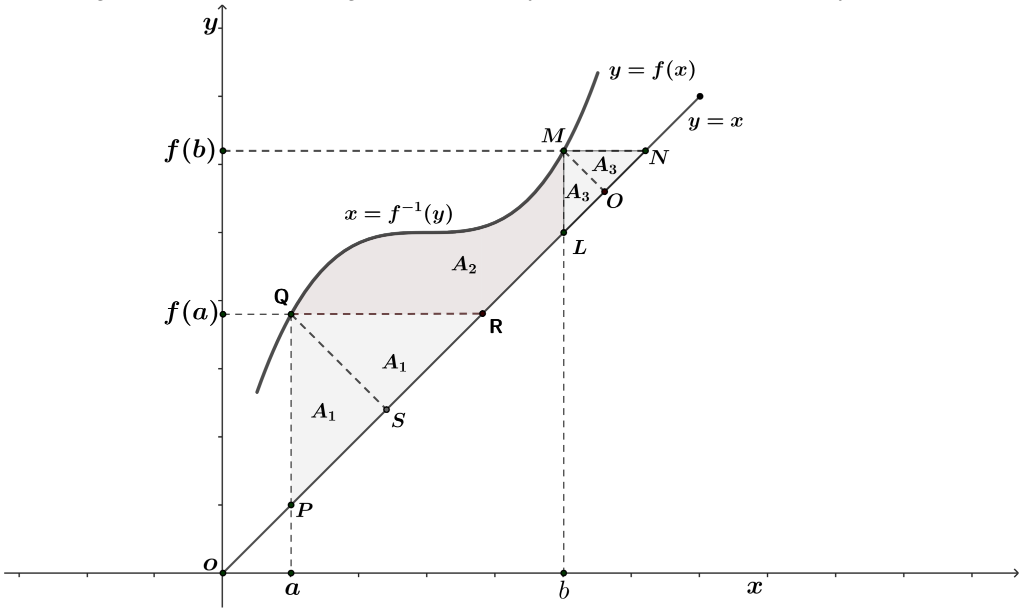
Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and derivable, then the area limited by f, the line $y = x$, and the segments perpendicular to this line, such that $f(x) > x$ for all $x \in [a, b]$, that is, the line is entirely below the curve represented by f, (as suggested by Figure 3), that is given by :*

$$A = \frac{1}{2}\int_a^b (f(x) - x)(1 + f'(x))dx. \tag{3}$$

Although the formula for the area in question can be obtained by relating the area of the rectangles perpendicular to the line $y = x$ with base Δu to the area of the rectangles with base Δx, as illustrated in the Figure 3. We can choose a geometric demonstration that explores the nature of the curves f and f^{-1}.

Proof. *Initially, we construct horizontal segments ($\overline{QR}$ and $\overline{MN}$) and vertical segments ($\overline{QP}$ and $\overline{ML}$) connecting the graph of f with the identity function.*

Figure 4 – Area of a region bounded by a function and the identity function



Source: Elaborated by the authors.

According to the Figure 4, it follows that triangle PQR is isosceles because ($\mid \overline{QP} \mid = \mid \overline{QR} \mid$). Indeed,

$$\mid \overline{QP} \mid = f(a) - a,$$
$$\mid \overline{QR} \mid = f(a) - a.$$

Therefore,

$$\mid \overline{QP} = \mid \overline{QR} \mid .$$

A similar statement can be made for triangle LMN. On the other hand, the height ($\overline{QS}$) divides it into two triangles of equal area, which we will denote as A_1. An equivalent statement regarding the height ($\overline{MO}$); consequently, the two triangles contained within triangle LMN have the same area, which we will denote as A_3. Therefore, the area in question will be denoted by A is given by the expression $A_1 + A_2 + A_3$, as shown in Figure 4.

In this way,

$$\int_a^b (f(x) - x)dx = 2A_1 + A_2. \tag{4}$$

By integration around the y-axis, we have:

$$\int_{f(a)}^{f(b)} (y - f^{-1}(y))dy = 2A_3 + A_2. \tag{5}$$

Adding Eq. (4) and Eq. (5) , we obtain:

$$\int_a^b (f(x) - x)dx + \int_{f(a)}^{f(b)} (y - f^{-1}(y))dy = 2(A_1 + A_2 + A_3). \tag{6}$$

Since $x = f^{-1}(y)$, then we have

$$\int_{f(a)}^{f(b)} (y - f^{-1}(y))dy = \int_a^b f'(x)(f(x) - x)dx. \tag{7}$$

Thus,

$$2A = \int_a^b [(f(x) - x)(1 + f'(x))]dx. \tag{8}$$

Therefore,

$$A = \frac{1}{2}\int_a^b [(f(x) - x)(1 + f'(x))]dx.$$

3 Conclusion

In both of the properties addressed here, pragmatism in their demonstration became evident, highlighting the advantages of exploring the geometric aspect of the concepts involved in these properties, as well as issues of symmetry. We believe that alternating these types of demonstrations with the traditionally analytical approaches greatly enriches the learning experience. Note that in the demonstration of Theorem 2.1 it was not necessary to assume that the derivative of the function is continuous. Furthermore, it is interesting to note that the relationship given by Theorem 2.1 can be used to calculate the definite integral of f/f^{-1} if the integral of f^{-1}/f is more difficult to calculate, as is the case of the integral $\int_1^e \ln x dx$. On the other hand, in the case of the question that leads us to the second result, it is possible to raise the question of obtaining the volume of the solid obtained by rotating this region around the line $y = x$. We believe that alternating these types of demonstrations with the traditionally analytical approaches greatly enriches the learning experience.

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