On non strictly diagonally dominant pentadiagonal matrices

Sobre matrizes pentadiagonais não estritamente diagonais dominantes

Matrices pentadiagonales que no son estrictamente diagonalmente dominantes

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Abstract: Based on Crout's method, we will present, in this work, new non singularity criteria and sufficient conditions for existence of the LU factorization, for non strictly diagonally dominant pentadiagonal matrices. Crout's method is a recursive process of n stages that obtains the factorization $A = LU$ of a pentadiagonal matrix of order n. In this recursive process of obtaining both the lower triangular matrix L and the upper triangular matrix U, the parameters α_i, $1 \leq i \leq n$, must be non-zero to ensure that $\det(A) \neq 0$ and $A = LU$. Crout's recursive method is replaced by the analysis of sufficient conditions that can be verified simultaneously with low computational cost.

Keywords: Crout's method; pentadiagonal matrix; non strictly diagonally dominant matrices.

Resumo: Baseados no método de Crout, nós apresentaremos neste trabalho novos critérios de não singularidade, e de existência de fatoração LU, para matrizes pentadiagonais não estritamente diagonais dominantes. O método de Crout é um processo recursivo de n estágios que obtém a fatoração $A = LU$ de uma matriz pentadiagonal de ordem n. Nesse processo recursivo de obtenção tanto da matriz triangular inferior L, quanto da matriz triangular superior U, os parâmetros α_i, $1 \leq i \leq n$, devem ser não nulos para garantir que $\det(A) \neq 0$ e que $A = LU$. Em nosso trabalho, o método recursivo de Crout é substituído pela análise de condições suficientes que podem ser verificadas simultaneamente, com baixo custo computacional.

Palavras-chave: método de Crout; matriz pentadiagonal; matrizes não estritamente diagonais dominantes.

Resumen: Basados en el método de Crout, presentaremos en este trabajo nuevos criterios de no singularidad y de existencia de factorización LU para matrices pentadiagonales no estrictamente dominantes en la

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diagonal. El método de Crout es un proceso recursivo de n etapas que obtiene la factorización $A = LU$ de una matriz pentadiagonal de orden n. En este proceso recursivo para obtener tanto la matriz triangular inferior L como la matriz triangular superior U, los parámetros α_i, $1 \leq i \leq n$, deben ser no nulos para asegurar que $\det(A) \neq 0$ y que $A = LU$. En nuestro trabajo, el método recursivo de Crout es sustituido por el análisis de condiciones suficientes que pueden ser verificadas simultáneamente, con bajo costo computacional.

Palabras clave: método de Crout; matriz pentadiagonal; matrices no estrictamente dominantes en la diagonal.

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1 Introduction

The development of methods for characterization of inverses of tridiagonal and pentadiagonal matrices has been studied by many authors, some of them will be commented on the text.

Fischer and Usmani (1969) gave a general analytical formula for inverse of symmetric Toeplitz tridiagonal matrices. The research objective was to obtain accurate error bounds for some finite difference approximations to 2-point boundary value problems.

Meurant (1992) presented a good review on research concerning the characterization of inverses of symmetric tridiagonal and block tridiagonal matrices as well as results concerning the decay of the elements of the inverses. According to this author, closed form explicit formulas for elements of the inverses can only be given for special matrices, e. g., Toeplitz tridiagonal matrices corresponding, for instance, to constant coefficients 1D elliptic partial differential equations (PDE), or for block matrices arising from separable 2D elliptic PDE (Bank; Rose, 1977).

There are also research focused on the development of algorithms for finding the inverse of any general non-singular tridiagonal or pentadiagonal matrix. If you want to know more on this matter, we recommend the articles of El-Mikkawy (2004), and Zhao and Huang (2008). The algorithms presented in those papers are suited for implementation using computer algebra systems such as MAPLE, MATHEMATICA, and MATLAB, for example.

The results of those research usually depend on the existence of the LU factorization of a non-singular matrix A, such that $A = LU$. Besides, many articles assume that the matrix is invertible or present non trivial conditions that ensure the non singularity of A and its LU factorization.

In 2023, we have developed a low-cost test for detecting in a simple way when a diagonally dominant pentadiagonal matrix is non-singular and has an LU decomposition. Remember that A of order n is a diagonally dominant matrix if $|A_{ii}| \geq |A_{i1}| + |A_{i2}| + \cdots + |A_{in}|$, $\forall i, 1 \leq i \leq n$. In this case, we say that A is a non strictly diagonally dominant matrix. This result was published in Proceeding Series of the Brazilian Society of Computational and Applied Mathematics (Almeida; Remigio, 2023).

We cite three works concerning these issues: Almeida and Remigio (Mar. 2023); Johnson, Marijuán and Pisonero (2023), and Kolotilina (2003). In the first paper, the authors have present a sufficient condition for existence of the LU factorization of a Toeplitz symmetric tridiagonal matrix A. They used an analysis based on the parameters of Crout's method, and concluded that $\det(A) \neq 0$. In the second paper, the authors have characterized, in terms of combinatorial structure and sign pattern, when a weakly (non strictly) diagonally dominant matrix may be invertible. In the third paper, the author has presented necessary and sufficient conditions for non singularity of the non strictly block diagonally dominant matrices.

Based on the results of Almeida and Remigio (2023), we are going to present in our work simple sufficient conditions for non singularity, and existence of the LU factorization, of non strictly diagonally dominant pentadiagonal (or tridiagonal) matrices. These conditions are simple because they can be verified simultaneously and with low operational cost and do not require the computationally expensive calculations of recursive processes like the Crout's method (which will be presented in next Section 2) and methods based on determinants.

Finally, the paper is organized as follows. In Section 2 some definitions are presented as well as preliminary results which will be used in later sections. We present the set P_D which is a subset of diagonally dominant pentadiagonal matrices, and we present sufficient conditions for non singularity, and existence of LU factorization, of diagonally dominant pentadiagonal matrices (Definition 2.1). In Section 3 we improve the results obtained in Section 2 by presenting two new theorems. In Section 4, it can be seen the results that relate the principal minors of a pentadiagonal diagonally dominant matrix to its parameters from Crout's decomposition. We prove the following strong result (Theorem 4.5): "Let A be a pentadiagonal diagonally dominant matrix; if $\det(A) \neq 0$, then $A = LU$, that is, there is the Crout's decomposition of matrix A". In Section 5 we define a reverse-permuted matrix. Based on this definition we can extend the results obtained in Sections 2 and 3. In Section 6 we present the conclusions of the work.

2 Definitions and preliminary results

Following the notation given in Almeida and Remigio (2023), $M_{n\times n}(\mathbb{R})$ represents the set of all matrices of order n with elements in $\mathbb{R}$. In this work, we consider $A = (A_{ij}) \in M_{n\times n}(\mathbb{R})$, with $A_{ij} = 0$ whenever $|i - j| > 2$ (pentadiagonal matrix). This matrix is represented by:

$$A = \begin{pmatrix} d_1 & a_1 & c_1 & 0 & 0 & 0 & \dots & & & & & 0 \\ b_2 & d_2 & a_2 & c_2 & 0 & 0 & & & & & & \vdots \\ e_3 & b_3 & d_3 & a_3 & c_3 & 0 & & & & & & \\ 0 & e_4 & b_4 & d_4 & a_4 & c_4 & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \\ & & & & & & & \ddots & \ddots & & & \\ \vdots & & & & & & & & & & & \\ 0 & & & & & & & e_{n-2} & b_{n-2} & d_{n-2} & a_{n-2} & c_{n-2} \\ & & & & & & & & e_{n-1} & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & \dots & & & & & & & & e_n & b_n & d_n \end{pmatrix}. \tag{1}$$

In this case, we consider $b_1 = e_1 = e_2 = c_{n-1} = c_n = a_n = 0$. As we are interested in studying non strictly diagonally dominant pentadiagonal matrices, we also consider $|d_i| \geq |e_i| + |b_i| + |a_i| + |c_i|$, for all i, $1 \leq i \leq n$. If, $|d_i| > |e_i| + |b_i| + |a_i| + |c_i|$, for all i, then A is a strictly diagonally dominant matrix. In numerical analysis, it is well known that every strictly diagonally dominant matrix is non-singular and has LU decomposition.

Our goal is to find sufficient conditions that guarantee that a diagonally dominant pentadiagonal matrix is non-singular and has LU decomposition. To do this, we will consider the set P_D which is composed by pentadiagonal matrices that are diagonally dominant and that satisfy the conditions shown in the definition below.

Definition 2.1. *The set P_D is defined as the set of pentadiagonal matrices A* (1) *such that their diagonal elements satisfy: $d_i \neq 0$, $|d_i| \geq |e_i| + |b_i| + |a_i| + |c_i|, i \in \{1, \dots, n\}$. Besides, the elements on each row of A must satisfy one of the following conditions: (a) $b_i = e_i = 0$; or*

(b) $|d_i| > |e_i| + |b_i| + |a_i| + |c_i|$; or

(c) $b_i^2 + e_i^2 \neq 0$, $|d_i| = |e_i| + |b_i| + |a_i| + |c_i|$, and $a_i^2 + c_i^2 \neq 0$; or

(d) $b_i \neq 0$, $a_i = c_i = 0$, $|d_i| = |e_i| + |b_i| + |a_i| + |c_i|$, and $|d_{i-1}| > |e_{i-1}| + |b_{i-1}| + |a_{i-1}| + |c_{i-1}|$; or

(e) $e_i \neq 0$, $a_i = c_i = 0$, $|d_i| = |e_i| + |b_i| + |a_i| + |c_i|$, *and* $|d_{i-2}| > |e_{i-2}| + |b_{i-2}| + |a_{i-2}| + |c_{i-2}|$*; or*

(f) $e_i = 0$, $b_i \neq 0$, $b_{i-1} = 0$, $a_{i-1} \neq 0$, $c_{i-1} = 0$ *and* $sgn(b_i.d_i) = -sgn(a_{i-1}.d_{i-1})$.

Remark: Every strictly diagonally dominant pentadiagonal matrix belongs to the set P_D, according to the item (b) from Definition 2.1.

Let A be a pentadiagonal matrix as shown in (1). According Zhao and Huang (2008), if $A = LU$, then L and U are pentadiagonal matrices given by

$$L = \begin{bmatrix} \alpha_1 & 0 & & & \dots & 0 \\ \beta_2 & \alpha_2 & 0 & & & \vdots \\ e_3 & \beta_3 & \alpha_3 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & & & 0 \\ 0 & \dots & & e_n & \beta_n & \alpha_n \end{bmatrix}, \quad U = \begin{bmatrix} 1 & \gamma_1 & \epsilon_1 & 0 & \dots & 0 \\ 0 & 1 & \gamma_2 & \epsilon_2 & 0 & \vdots \\ & 0 & 1 & \gamma_3 & \epsilon_3 & \\ & & & \ddots & \ddots & \\ \vdots & & & & 1 & \gamma_{n-1} \\ 0 & \dots & & & 0 & 1 \end{bmatrix}, \text{where} \tag{2}$$

$$\alpha_i = \begin{cases} d_1, \;\; i = 1; \;\; d_2 - \gamma_1\beta_2, & i = 2; \\ d_i - \gamma_{i-1}\beta_i - \epsilon_{i-2}e_i, & i \in \{3, 4, \dots, n\}; \end{cases} \tag{3}$$

$$\gamma_i = \begin{cases} a_1/\alpha_1, & i = 1; \\ \dfrac{a_i - \epsilon_{i-1}\beta_i}{\alpha_i}, & i \in \{2, 3, \dots, n-1\}; \end{cases} \tag{4}$$

$$\epsilon_i = \left\{ \; \frac{c_i}{\alpha_i}, \quad i \in \{1, 2, \dots, n-2\}; \right. \tag{5}$$

$$\beta_i = \begin{cases} b_2, & i = 2; \\ b_i - \gamma_{i-2}e_i, & i \in \{3, 4, \dots, n\}. \end{cases} \tag{6}$$

The previous method is known as Crout's decomposition. This decomposition is possible whenever $\alpha_i \neq 0,\ 1 \leq i \leq n$ and, in this case, $\det(A) \neq 0$.

Remark: Based on Crout's decomposition, the Theorem 3.1 from Almeida and Remigio (2023), states that if $A \in P_D$, then $A = LU$, and $\det(A) \neq 0$.

As is to be expected, there are pentadiagonal diagonally dominant matrices that do not satisfy the sufficient conditions presented in Definition 2.1. For these matrices, the non singularity criteria based on Crout's method fail. We next present an example of this type.

Example 1. The following matrix $A \notin P_D$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that the row 2 does not satisfy the conditions presented in Definition 2.1. Therefore, our test on non singularity is inconclusive, since the sufficient conditions have not been met. Although, we know $\det(A) = 0$ because the matrix has two equal rows.

Remark: In previous example, if the second row was $1 \quad -1 \quad 0 \quad 0 \quad 0$, then the matrix would belong to P_D, because $b_1 = e_1 = 0$; $e_2 = 0$, $b_1 = 0$ and $sgn(b_2.d_2) = -sgn(a_1.d_1)$; $|d_3| > |e_3|+|b_3|+|a_3|+|c_3| = 0$; $|d_4| > |e_4|+|b_4|+|a_4|+|c_4| = 0$ and $|d_5| > |e_5|+|b_5|+|a_5|+|c_5| = 0$. Therefore, in this case, $\det(A) \neq 0$ and A would have an LU decomposition.

It is well known that if A is a squared matrix of order n which has an LU decomposition, where $L_{ii} \neq 0$ and $U_{ii} = 1$, $1 \leq i \leq n$, then A^T also has an $\mathcal{LU}$ decomposition, where $\mathcal{L}_{ii} \neq 0$ and $\mathcal{U}_{ii} = 1$, $1 \leq i \leq n$.

Remark: Considering the previous result, the subjects of our studies are pentadiagonal matrices A, such that A, or A^T belongs to the set P_D.

We next present an example concerning the previous remark.

Example 2. The following matrix A is not diagonally dominant (observe the first and third rows of A).

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 1 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 3 & 5 & 1 & -1 & 0 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

However, its transpose A^T is diagonally dominant. Besides, note that rows 1 to 5 from the transpose matrix satisfy the conditions presented in Definition 2.1: row 1 – item (a); rows $2, 3$ and 4 – item (c); row 5 – item (b). Thus, by Theorem 3.1 from Almeida and Remigio (2023), A^T belongs to set P_D. Therefore, A has an LU decomposition, and $\det(A) = \det(A^T) \neq 0$.

We finish this section by stating a lemma that will be used in the next section. This lemma was proved by Almeida and Remigio (2023, Lemma 2.3).

Lemma 2.2. *Let a, b, c, d, and e be real numbers that satisfy: $|d| \geq |a| + |b| + |c| + |e|$, and $d \neq 0$. Suppose the real numbers β, γ, ϵ, $\tilde{\gamma}$, $\tilde{\epsilon}$, α are such that: (i) $\beta = b - \tilde{\gamma}e$, (ii) $|\gamma| + |\epsilon| \leq 1$, (iii) $|\tilde{\gamma}| + |\tilde{\epsilon}| \leq 1$, and (iv) $\alpha = d - \gamma\beta - \tilde{\epsilon}e \neq 0$. Then (I) $\dfrac{|a - \epsilon\beta| + |c|}{|\alpha|} \leq 1$; (II) $|d| > |a| + |b| + |c| + |e| \longrightarrow \dfrac{|a - \epsilon\beta| + |c|}{|\alpha|} < 1$; (III) $e^2 + b^2 \neq 0$, $|\gamma| + |\epsilon| < 1$, $|\tilde{\gamma}| + |\tilde{\epsilon}| < 1 \longrightarrow \dfrac{|a - \epsilon\beta| + |c|}{|\alpha|} < 1$; (IV) $e \neq 0$, $|\tilde{\gamma}| + |\tilde{\epsilon}| < 1 \longrightarrow \dfrac{|a - \epsilon\beta| + |c|}{|\alpha|} < 1$.*

3 Two new non singularity criteria for non strictly diagonally dominant pentadiagonal matrices

The next two theorems provide further conditions for a pentadiagonal diagonally dominant matrix A to have Crout's decomposition, and $\det(A) \neq 0$.

Theorem 3.1. *Let A be a pentadiagonal diagonally dominant matrix as shown in* (1)*. Consider* (3)*,* (4)*,* (5)*,* (6) *and suppose there is an integer $k > 1$ such that $\alpha_i \neq 0$, $1 \leq i \leq k-1$, $|d_{k-1}| > |e_{k-1}| + |b_{k-1}| + |a_{k-1}| + |c_{k-1}|$, and $b_{k+2j} \neq 0$ and $e_{k+(2j+1)} \neq 0$, whenever $j \geq 0$ and $k + (2j+1) \leq n$. Observe that $b_{k+2j} = b_n \neq 0$ and $k + (2j+1) = n+1$, if $n-k$ is an even number and $j = (n-k)/2$. On the other hand, if $n-k$ is odd and $j = (n-k-1)/2$, then $e_{k+(2j+1)} = e_n \neq 0$ and $k + 2j = n-1$. Thus, $\alpha_i \neq 0$, $1 \leq i \leq n$. Therefore, $A = LU$ and $\det(A) \neq 0$.*

Proof. *Note that $\alpha_i \neq 0, \forall i$, $1 \leq i \leq k-1 \longrightarrow |\gamma_i| + |\epsilon_i| \leq 1$, $\forall i$, $1 \leq i \leq k-1$, because $|d_i| \geq |e_i| + |b_i| + |a_i| + |c_i|$. The demonstration is based on mathematical induction and Lemma 2.2 (item I).*

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Considering Lemma 2.2, item (II), $|\gamma_{k-1}| + |\epsilon_{k-1}| < 1$. *If* $k = 2$, *then* $|\alpha_2| = |d_2 - \beta_2\gamma_1| = |d_2 - \gamma_1 b_2| \geq |d_2| - |\gamma_1||b_2| > |d_2| - |b_2| \geq |a_2| + |c_2|$, *because* $b_2 \neq 0$. *Thus,* $|\gamma_2| + |\epsilon_2| \leq 1$, *according to Lemma 2.2, item (I). If* $k > 2$, *then, according to Lemma 2.2 and the theorem's hypotheses, we obtain that* $|\gamma_{k-1}||\gamma_{k-2}| \leq |\gamma_{k-2}|$. *Hence,* $|\gamma_{k-1}||\gamma_{k-2}| + |\epsilon_{k-2}| \leq |\gamma_{k-2}| + |\epsilon_{k-2} \leq 1$. *Thus,* $|\alpha_k| = |d_k - \beta_k\gamma_{k-1} - \epsilon_{k-2}e_k| \geq |d_k| - |\gamma_{k-1}||\beta_k| - |\epsilon_{k-2}||e_k| \geq |d_k| - |\gamma_{k-1}|(|b_k| + |\gamma_{k-2}||e_k|) - |\epsilon_{k-2}||e_k| > |d_k| - |b_k| - |e_k| \geq |a_k| + |c_k| \geq 0$, *because* $b_k \neq 0$. *Therefore,* $|\alpha_k| > 0$ *and, according to Lemma 2.2, item (I),* $|\gamma_k| + |\epsilon_k| = |\frac{a_k - \epsilon_{k-1}\beta_k}{\alpha_k}| + |\frac{c_k}{\alpha_k}| \leq 1$.

Now, note that $e_{k+1} \neq 0$ *and* $|\gamma_k||\gamma_{k-1}| \leq |\gamma_{k-1}|$. *Hence,* $|\gamma_k||\gamma_{k-1}| + |\epsilon_{k-1}| \leq |\gamma_{k-1}| + |\epsilon_{k-1} < 1$. *Thus,* $|\alpha_{k+1}| = |d_{k+1} - \beta_{k+1}\gamma_k - \epsilon_{k-1}e_{k+1}| \geq |d_{k+1}| - |\gamma_k||\beta_{k+1}| - |\epsilon_{k-1}||e_{k+1}| \geq |d_{k+1}| - |\gamma_k||b_{k+1}| - (|\gamma_k||\gamma_{k-1}| + |\epsilon_{k-1}|)|e_{k+1}| > |d_{k+1}| - |b_{k+1}| - |e_{k+1}| = |a_{m+1}| + |c_{m+1}| \geq 0$. *In this way,* $|\alpha_{k+1}| > 0$ *and, according to Lemma 2.2, item (IV),* $|\gamma_{k+1}| + |\epsilon_{k+1}| = |\frac{a_{k+1} - \epsilon_k\beta_{k+1}}{\alpha_{k+1}}| + |\frac{c_{k+1}}{\alpha_{k+1}}| < 1$.

In order to prove by induction, suppose that $|\gamma_{k+2j}| + |\epsilon_{k+2j}| \leq 1$, $|\gamma_{k+(2j+1)}| + |\epsilon_{k+(2j+1)}| < 1$, $\alpha_{k+2j} \neq 0$, *and* $\alpha_{k+(2j+1)} \neq 0$, $\forall j, 0 \leq j \leq m$.

If $M = k + 2(m+1)$, *then* $|\gamma_{M-1}||\gamma_{M-2}| \leq |\gamma_{M-2}|$. *Hence,* $|\gamma_{M-1}||\gamma_{M-2}| + |\epsilon_{M-2}| \leq |\gamma_{M-2}| + |\epsilon_{M-2} \leq 1$. *By using induction hypothesis, if* $b_M \neq 0$, *then*

$$|\alpha_M| = |d_M - \beta_M\gamma_{M-1} - \epsilon_{M-2}e_M| \geq |d_M| - |\gamma_{M-1}||\beta_M| - |\epsilon_{M-2}||e_M| \geq$$

$$|d_M| - |\gamma_{M-1}|(|b_M| + |\gamma_{M-2}||e_M|) - |\epsilon_{M-2}||e_M| > |d_M| - |b_M| - |e_M| \geq |a_M| + |c_M \geq 0.$$

Therefore, $|\alpha_M| > 0$ *and, according to Lemma 2.2, item (I),* $|\gamma_M| + |\epsilon_M| = |\frac{a_M - \epsilon_{M-1}\beta_M}{\alpha_M}| + |\frac{c_M}{\alpha_M}| \leq 1$. *Moreover,* $|\gamma_M||\gamma_{M-1}| \leq |\gamma_{M-1}|$ *and, by the induction hypothesis,* $|\gamma_M||\gamma_{M-1}| + |\epsilon_{M-1}| \leq |\gamma_{M-1}| + |\epsilon_{M-1}| < 1$.

If $e_{M+1} \neq 0$, *then*

$$|\alpha_{M+1}| = |d_{M+1} - \beta_{M+1}\gamma_M - \epsilon_{M-1}e_{M+1}| \geq$$

$$|d_{M+1}| - |\gamma_M||\beta_{M+1}| - |\epsilon_{M-1}||e_{M+1}| \geq$$

$$|d_{M+1}| - |\gamma_M|(|b_{M+1}| + |\gamma_{M-1}||e_{M+1}|) - |\epsilon_{M-1}||e_{M+1}| \geq$$

$$|d_{M+1}| - |\gamma_M||b_{M+1}| - (|\gamma_M||\gamma_{M-1}| + |\epsilon_{M-1}|)|e_{M+1}| >$$

$$|d_{M+1}| - |b_{M+1}| - |e_{M+1}| \geq |a_{M+1}| + |c_{M+1} \geq 0.$$

In this way, $|\alpha_{M+1}| > 0$ and, according to Lemma 2.2, item (IV), $|\gamma_{M+1}| + |\epsilon_{M+1}| = |\frac{a_{M+1} - \epsilon_M \beta_{M+1}}{\alpha_{M+1}}| + |\frac{c_{M+1}}{\alpha_{M+1}}| < 1.$

Therefore, by mathematical induction, it is possible to conclude that $\alpha_i \neq 0,\ 1 \leq i \leq n.$ □

Theorem 3.2. *Let A be a pentadiagonal diagonally dominant matrix as shown in* (1)*. Consider* (3)*,* (4)*,* (5)*,* (6) *and suppose there is an integer $k > 2$ such that $\alpha_i \neq 0,\ 1 \leq i \leq k-1$, $|d_{k-2}| > |e_{k-2}| + |b_{k-2}| + |a_{k-2}| + |c_{k-2}|$, $|d_{k-1}| > |e_{k-1}| + |b_{k-1}| + |a_{k-1}| + |c_{k-1}|$ and $b_{k+j}^2 + e_{k+j}^2 \neq 0$, whenever $j \geq 0$ and $k + j \leq n$. Thus, $\alpha_i \neq 0,\ 1 \leq i \leq n$. Therefore, $A = LU$ and $\det(A) \neq 0$.*

Proof. *The demonstration follows the same steps of Theorem 3.1.* □

We next present two examples of applications of Theorem 3.1 and Theorem 3.2.

Example 3 (Theorem 3.1). Matrix A given below has Crout's decomposition and $\det(A) \neq 0$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 1/2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

The first row elements of matrix A satisfy the item (a), and the second row elements satisfy the item (b) described in Definition 2.1. Therefore, we can apply Theorem 3.1 with $k = 3$ to obtain the desired result. It would not be possible to obtain the result just using the conditions described in Definition 2.1, without applying the theorem. Note that the third row elements of matrix A satisfy the item (d), and the forth row elements of matrix A satisfy the item (e) described in Definition 2.1. However, the fifth row of A does not satisfy none of the six items described in that definition.

Example 4 (Theorem 3.2). Matrix A given below has Crout's decomposition and $\det(A) \neq 0$.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 1/2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The first and the second row elements of matrix A satisfy the item (b) described in Definition 2.1. Therefore, we can apply Theorem 3.2 with $k = 3$ to obtain the desired result.

4 Relation between principal minors and parameters of the Crout's decomposition

A well known result of numerical analysis says that any matrix A, of order n, has LU decomposition if all principal minors of order k are non-zero, $1 \leq k \leq n-1$. A principal minor of order k is the determinant of the submatrix formed by the first k rows and first k columns of matrix A. In this section we are going to prove the following strong result (Theorem 4.5) "Let A be a pentadiagonal diagonally dominant matrix as shown in (1). If $\det(A) \neq 0$, then $A = LU$, that is, there is the Crout's decomposition of matrix A".

In order to reach the objective proposed previously, we will present two theorems regarding the relationship between principal minors from a pentadiagonal diagonally dominant matrix and the parameters of the Crout's decomposition given in (3), (4), (5) and (6). The results from these theorems are also valid for tridiagonal matrices just considering $e_i = c_i = 0,\ 1 \leq i \leq n$, in (1).

Firstly, we are going to present the following particular cases. The principal minor of order 1 from a pentadiagonal diagonally dominant matrix is given by $M_1 = d_1 = \alpha_1$, and the one of order 2 is given by $M_2 = d_1\, d_2 - a_1\, b_2$. If $\alpha_1 \neq 0$, then $M_1/\alpha_1 = 1$ and $M_2/\alpha_1 = d_2 - \gamma_1 b_2 = \alpha_2$. The Laplace Expansion by the third row results in $M_3 = d_3 M_2 - b_3(a_2 d_1 - c_1 b_2) + e_3(a_2 a_1 - c_1 d_2)$. In this way, if $\alpha_i \neq 0, 1 \leq i \leq 2$, then $M_1/\alpha_1 = 1$, $M_2/(\alpha_1\alpha_2) = 1$, and $M_3/(\alpha_1\alpha_2) = d_3 - b_3[(a_2 - \epsilon_1 b_2)/\alpha_2] + e_3[(\gamma_1 a_2 - \epsilon_1 d_2)/\alpha_2]$. Besides, by (4), we obtain $a_2 = \gamma_2\alpha_2 + \epsilon_1 b_2$. Hence, $M_3/(\alpha_1\alpha_2) = d_3 - b_3\gamma_2 + e_3(\gamma_1\gamma_2) - e_3[\epsilon_1(d_2 - \gamma_1 b_2)/\alpha_2]$. Therefore,

$$M_3/(\alpha_1\alpha_2) = d_3 - \gamma_2(b_3 - \gamma_1 e_3) - \epsilon_1 e_3 = d_3 - \gamma_2\beta_3 - \epsilon_1 e_3 = \alpha_3.$$

The following notation $\mathbf{M_{i,j}}$ will be used to indicate the matrix of order $i-1$ resulting from the removal of row i and column j from the matrix corresponding to the principal minor M_i. In this way, given the matrices $\mathbf{M_{i,i-1}}$ and $\mathbf{M_{i,i-2}}$, consider Laplace Expansion by the last column of these matrices to calculate their determinants.

To obtain the determinant of $\mathbf{M_{i,i-1}}$, it will be necessary to calculate two other determinants from matrices of order $i-2$. These matrices will be represented by $B^{(i-2)}$ and $B^{(i-1)}$. The matrix

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$B^{(i-2)}$ is resulting from the removal of column $i-1$ and row $i-1$ from the matrix corresponding to principal minor M_{i-1}. Thus, $\det(B^{(i-2)}) = M_{i-2}$. To obtain the matrix $B^{(i-1)}$, the last row of $B^{(i-2)}$ should be replaced by the last row of the matrix corresponding to principal minor M_{i-1}, without the element from column $i-1$.

To obtain the determinant of $\mathbf{M_{i,i-2}}$, it will be necessary to calculate two other determinants from matrices of order $i-2$. These matrices will be represented by $E^{(i-2)}$ e $E^{(i-1)}$. The matrix $E^{(i-2)}$ is resulting from the removal of column $i-2$ and row $i-1$ from the matrix corresponding to principal minor M_{i-1}. To obtain the matrix $E^{(i-1)}$, the last row of $E^{(i-2)}$ should be replaced by the last row of the matrix corresponding to principal minor M_{i-1}, without the element from column $i-2$.

In the next **Example 5**, we consider a particular case of those matrices mentioned above.

Example 5: Respecting previous notation and considering $i = 4$, we obtain that,

$$M_4 = \begin{vmatrix} d_1 & a_1 & c_1 & 0 \\ b_2 & d_2 & a_2 & c_2 \\ e_3 & b_3 & d_3 & a_3 \\ 0 & e_4 & b_4 & d_4 \end{vmatrix} = d_4\, M_3 - b_4\, \det(\mathbf{M_{4,3}}) + e_4\, \det(\mathbf{M_{4,2}}).$$

Therefore,

$$M_4 = d_4 \begin{vmatrix} d_1 & a_1 & c_1 \\ b_2 & d_2 & a_2 \\ e_3 & b_3 & d_3 \end{vmatrix} - b_4 \begin{vmatrix} d_1 & a_1 & 0 \\ b_2 & d_2 & c_2 \\ e_3 & b_3 & a_3 \end{vmatrix} + e_4 \begin{vmatrix} d_1 & c_1 & 0 \\ b_2 & a_2 & c_2 \\ e_3 & d_3 & a_3 \end{vmatrix}.$$

Note that:

$$\det(\mathbf{M_{4,3}}) = a_3\, \det(B^{(2)}) - c_2\, \det(B^{(3)}) = a_3 \begin{vmatrix} d_1 & a_1 \\ b_2 & d_2 \end{vmatrix} - c_2 \begin{vmatrix} d_1 & a_1 \\ e_3 & b_3 \end{vmatrix},$$

$$\det(\mathbf{M_{4,2}}) = a_3\, \det(E^{(2)}) - c_2\, \det(B^{(3)}) = a_3 \begin{vmatrix} d_1 & c_1 \\ b_2 & a_2 \end{vmatrix} - c_2 \begin{vmatrix} d_1 & c_1 \\ e_3 & d_3 \end{vmatrix}.$$

As seen previously, if $\alpha_i \neq 0,\ 1 \leq i \leq 3$, then $\alpha_1 = d_1 = M_1,\quad M_1/\alpha_1 = 1,\quad \gamma_1 = a_1/d_1,$ $\epsilon_1 = c_1/\alpha_1,\ \ \alpha_2 = d_2 - \gamma_1\beta_2 = d_2 - \gamma_1 b_2,\ \ \det(B^{(2)})/(\alpha_1\alpha_2) = M_2/(\alpha_1\alpha_2) = 1,\ \ \gamma_2 = (a_2 - \epsilon_1\beta_2)/\alpha_2,$ $\beta_3 = b_3 - \gamma_1 e_3,\ \ \alpha_3 = d_3 - \gamma_2\beta_3 - \epsilon_1 e_3,$ and $M_3/(\alpha_1\alpha_2\alpha_3) = 1$. Additionally,

$c_2/\alpha_2 = \epsilon_2,\ \ \det(B^{(3)})/\alpha_1 = \beta_3,\ \ \det(E^{(2)})/(\alpha_1\alpha_2) = \gamma_2,$ and

$\det(E^{(3)})/\alpha_1 = d_3 - \epsilon_1 e_3 = \alpha_3 + \gamma_2\beta_3.$

Therefore,

$$\frac{M_4}{\alpha_1\alpha_2\alpha_3} = d_4 - b_4\frac{a_3 - \epsilon_2\beta_3}{\alpha_3} + e_4\frac{a_3\gamma_2 - \epsilon_2(\alpha_3 + \gamma_2\beta_3)}{\alpha_3} \longrightarrow$$

$$\frac{M_4}{\alpha_1\alpha_2\alpha_3} = d_4 - b_4\gamma_3 + \gamma_2\gamma_3 e_4 - \epsilon_2 e_4 = d_4 - \gamma_3\beta_4 - \epsilon_2 e_4 = \alpha_4,$$

where $\beta_4 = b_4 - \gamma_2 e_4$. □

The next theorem will be presented respecting previous notations.

Theorem 4.1. *Let A be a pentadiagonal diagonally dominant matrix as shown in* (1)*. Consider* (3)*,* (4)*,* (5)*,* (6) *and suppose there is an integer k, $1 < k \leq n$, such that $\alpha_i \neq 0, \forall i, 1 \leq i \leq k-1$. Then $\frac{M_i}{\alpha_1\alpha_2\cdots\alpha_i} = 1, \forall i, 1 \leq i \leq k-1$, and $\frac{M_k}{\alpha_1\alpha_2\cdots\alpha_{k-1}} = \alpha_k$. If $k = 1$ and $\alpha_0 = 1$, then $\frac{M_1}{\alpha_0} = M_1 = d_1 = \alpha_1$.*

Proof. *According to the beginning of this Section 4, the result is satisfied for k varying from 1 to 4. Besides, for $k = 4$, $\frac{\det(B^{(k-1)})}{\alpha_1\cdots\alpha_{k-3}} = \beta_{k-1}$, $\frac{\det(E^{(k-2)})}{\alpha_1\cdots\alpha_{k-2}} = \gamma_{k-2}$ and $\frac{\det(E^{(k-1)})}{\alpha_1\cdots\alpha_{k-3}} = \alpha_{k-1} + \gamma_{k-2}\beta_{k-1}$.*

In order to prove the result by induction, suppose that the previous equalities are valid for all positive integer k, $4 \leq k \leq m$, where $m \leq n$. Besides, $\frac{M_i}{\alpha_1\alpha_2\cdots\alpha_i} = 1, \forall i, 1 \leq i \leq k-1$, and $\frac{M_k}{\alpha_1\alpha_2\cdots\alpha_{k-1}} = \alpha_k$, for all k such that $4 \leq k \leq m$ and $\alpha_i \neq 0,\ 1 \leq i \leq k-1$.

Applying Laplace Expansion by the last row to calculate the principal minor M_{m+1}, we obtain that $M_{m+1} = d_{m+1}\, M_m - b_{m+1}\, \det(\mathbf{M_{m+1,m}}) + e_{m+1}\, \det(\mathbf{M_{m+1,m-1}})$. Now, using Laplace Expansion by the last column to calculate $\det(\mathbf{M_{m+1,m}})$ and $\det(\mathbf{M_{m+1,m-1}})$, we obtain $M_{m+1} = d_{m+1}\, M_m - b_{m+1}[a_m\, M_{m-1} - c_{m-1}\, \det(B^{(m)})] + e_i[a_m\, \det(E^{(m-1)}) - c_{m-1}\, \det(E^{(m)})]$.

Suppose that $\alpha_i \neq 0, \forall i, 1 \leq i \leq m$. According to the hypothesis of induction, $\frac{M_i}{\alpha_1\alpha_2\cdots\alpha_i} = 1,\ 1 \leq i \leq m$. Besides,

$$\frac{M_{m+1}}{\alpha_1\alpha_2\cdots\alpha_m} = d_{m+1} - b_{m+1}\frac{a_m - \epsilon_{m-1}\beta_m}{\alpha_m} + e_{m+1}\frac{a_m\gamma_{m-1} - \epsilon_{m-1}(\alpha_m + \gamma_{m-1}\beta_m)}{\alpha_m} \longrightarrow$$

$$\frac{M_{m+1}}{\alpha_1\alpha_2\cdots\alpha_m} = d_{m+1} - b_{m+1}\gamma_m + \gamma_{m-1}\gamma_m e_{m+1} - \epsilon_{m-1}e_{m+1} = d_{m+1} - \gamma_m\beta_{m+1} - \epsilon_{m-1}e_{m+1} = \alpha_{m+1}.$$

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Therefore, the result is valid for every integer $k, 1 \leq k \leq n.$ $\square$

Remark: For a tridiagonal matrix, the following result is valid: $M_i = d_i\, M_{i-1} - b_i\, a_{i-1}\, M_{i-2}$.

The next two lemmas will be useful in the proof of Theorem 4.4.

Lemma 4.2. *Let* A *be a pentadiagonal diagonally dominant matrix, as shown in* (1), *and* M_k *its principal minor of order* k, $1 \leq k < n$. *Consider the matrix corresponding to the principal minor* M_k *and its parameters of the Crout's decomposition. These parameters will be represented by the same notation used in* (3), (4), (5), (6), *but adding an overline to them. In this way, whenever* $\alpha_i \neq 0,\ \forall i,\ 0 \leq i \leq k-1$, *we will obtain* $\overline{\beta}_i = \beta_i,\ \overline{\alpha}_i = \alpha_i,\ \forall i,\ 1 \leq i \leq k;\ \ \overline{\gamma}_i = \gamma_i,\ \overline{\epsilon}_i = \epsilon_i,\ \forall i,\ 1 \leq i \leq k-2;$ $\overline{\gamma}_{k-1} = \gamma_{k-1},\ \overline{\epsilon}_{k-1} = 0;\ \overline{\gamma}_k = 0,\ \overline{\epsilon}_k = 0.$

Proof. *If* $k = 1$, *then* $\overline{\beta}_1 = 0 = \beta_1$ *(by convention, we are considering* $e_1 = b_1 = 0$*),* $\overline{\alpha}_1 = d_1 = \alpha_1$. *Note that* $\overline{\gamma}_1 = 0 = \overline{\epsilon}_1$, *because the elements* a_k *and* c_k *are not present in the matrix associated to the principal minor* M_k *and, for this case, conveniently, it is assumed that* $a_k = c_k = 0$.

If $k = 2$, *then* $\overline{\beta}_1 = 0 = \beta_1$, $\overline{\alpha}_1 = d_1 = \alpha_1$, $\overline{\gamma}_1 = a_1/d_1 = \gamma_1$ *and* $\overline{\epsilon}_1 = 0$. *Besides,* $\overline{\beta}_2 = b_2 = \beta_2$, $\overline{\alpha}_2 = d_2 - \overline{\gamma}_1\, b_2 = \alpha_2$, $\overline{\gamma}_2 = 0$, *and* $\overline{\epsilon}_2 = 0$.

If $k = 3$, *then* $\overline{\beta}_1 = 0 = \beta_1,\ \ \overline{\alpha}_1 = d_1 = \alpha_1,\ \ \overline{\gamma}_1 = a_1/d_1 = \gamma_1$ *and* $\overline{\epsilon}_1 = c_1/d_1 = \epsilon_1$. *Besides,* $\overline{\beta}_2 = b_2 = \beta_2,\ \ \overline{\alpha}_2 = d_2 - \overline{\gamma}_1\, b_2 = \alpha_2,\ \ \overline{\gamma}_2 = \frac{a_2 - \overline{\epsilon}_1 \overline{\beta}_2}{\overline{\alpha}_2} = \gamma_2$ *and* $\overline{\epsilon}_2 = 0$. *Additionally,* $\overline{\beta}_3 = b_3 - \overline{\gamma}_1\, e_3 = b_3 - \gamma_1\, e_3 = \beta_3,\ \ \overline{\alpha}_3 = d_3 - \overline{\gamma}_2\, \overline{\beta}_3 - \overline{\epsilon}_1\, e_3 = d_3 - \gamma_2\, \beta_3 - \epsilon_1\, e_3 = \alpha_3,\ \ \overline{\gamma}_3 = \overline{\epsilon}_3 = 0.$

Consider $k > 3$ *and suppose the conditions imposed in this theorem are valid for every* i, $1 \leq i < m \leq k-2$. *Then* $\overline{\beta}_m = b_m - \overline{\gamma}_{m-2}\, e_m = b_m - \gamma_{m-2}\, e_m = \beta_m,\ \overline{\alpha}_m = d_m - \overline{\gamma}_{m-1}\, \overline{\beta}_m - \overline{\epsilon}_{m-2}\, e_m = d_m - \gamma_{m-1}\, \beta_m - \epsilon_{m-2}\, e_m = \alpha_m,\ \ \overline{\gamma}_m = \frac{a_m - \overline{\epsilon}_{m-1} \overline{\beta}_m}{\overline{\alpha}_m} = \frac{a_m - \epsilon_{m-1}\, \beta_m}{\alpha_m} = \gamma_m$ *and* $\overline{\epsilon}_m = \frac{c_m}{\overline{\alpha}_m} = \epsilon_m$.

Therefore, by mathematical induction, the conditions are valid for every $i,\ 1 \leq i \leq k-2$. *Additionally, note that* $\overline{\beta}_{k-1} = b_{k-1} - \overline{\gamma}_{k-3}\, e_{k-1} = b_{k-1} - \gamma_{k-3}\, e_{k-1} = \beta_{k-1},\ \ \overline{\alpha}_{k-1} = d_{k-1} - \overline{\gamma}_{k-2}\, \overline{\beta}_{k-1} - \overline{\epsilon}_{k-3} = d_{k-1} - \gamma_{k-2}\, \beta_{k-1} - \epsilon_{k-3} = \alpha_{k-1},\ \ \overline{\gamma}_{k-1} = \frac{a_{k-1} - \overline{\epsilon}_{k-2} \overline{\beta}_{k-1}}{\overline{\alpha}_{k-1}} = \frac{a_{k-1} - \epsilon_{k-2}\, \beta_{k-1}}{\alpha_{k-1}} = \gamma_{k-1}$ *and* $\overline{\epsilon}_{k-1} = 0$. *Besides,* $\overline{\beta}_k = b_k - \overline{\gamma}_{k-2}\, e_k = b_k - \gamma_{k-2}\, e_k = \beta_k,\ \ \overline{\alpha}_k = d_k - \overline{\gamma}_{k-1}\, \overline{\beta}_k - \overline{\epsilon}_{k-2}\, e_k = d_k - \gamma_{k-1}\, \beta_k - \epsilon_{k-2}\, e_k = \alpha_k$, *and* $\overline{\gamma}_k = \overline{\epsilon}_k = 0$. *In this way, the lemma is proved.* $\square$

Lemma 4.3. *Let* A *be a pentadiagonal diagonally dominant matrix, as shown in* (1), *and consider the parameters of the Crout's decomposition* (3), (4), (5), (6). *If* $\alpha_i \neq 0, \forall i,\ 1 \leq i \leq j+1 \leq n$, *then, at the end of* j*-th stage of the Gaussian Elimination Method applied to matrix* A, *we will obtain the following elements: (Diagonal)* $d_i^{(j)} = \alpha_i,\ i \in \{1, \cdots, j+1\},\ \ d_{j+2}^{(j)} = d_{j+2} - \epsilon_j\, e_{j+2},\ d_i^{(j)} = d_i,\ \forall i,\ j+3 \leq i \leq n;$

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(First Upper Diagonal) $a_i^{(j)} = \gamma_i\,\alpha_i,\ i \in \{1,\cdots,j+1\},\ a_i^{(j)} = a_i,\ \forall i,\ j+2 \le i \le n$; *(Second Upper Diagonal)* $c_i^{(j)} = c_i,\ i \in \{1,\cdots,n\}$; *(First Lower Diagonal)* $b_i^{(j)} = 0,\ i \in \{1,\cdots,j+1\},\ b_{j+2}^{(j)} = \beta_{j+2}$, $b_i^{(j)} = b_i,\ \forall i,\ j+3 \le i \le n$; *(Second Lower Diagonal)* $e_i^{(j)} = 0,\ i \in \{1,\cdots,j+2\},\ e_i^{(j)} = e_i,\ \forall i,\ j+3 \le i \le n$.

Proof. *Let A be a pentadiagonal matrix of order n. Then, in each stage j of the Gaussian Elimination Method, we need to carry out elementary row operations to obtain null elements below the diagonal at the column j (elements $A_{j+1,j}$ and $A_{j+2,j}$, when $j < n-1$, and element $A_{n,n-1}$, when $j = n-1$). Additionally, in the row j of the pentadiagonal matrix A, the elements belonging to column $k > j+2$ are null. Therefore, in step j, $j < n-1$, the elements of the position $(j+1,j)$ and $(j+2,j)$ will become null and only elements from positions $(j+1,j+1)$ and $(j+1,j+2)$, $(j+2,j+1)$ and $(j+2,j+2)$ will be altered.*

In stage j, the multipliers are given by $m_{j+1}^{(j)} = b_{j+1}^{(j-1)}/d_j^{(j-1)}$, and $m_{j+2}^{(j)} = e_{j+2}^{(j-1)}/d_j^{(j-1)}$ (the superscript $j = 0$ represents the element from the original matrix). Therefore, in stage $j = 1$, $m_2^{(1)} = b_2/\alpha_1$, and $m_3^{(1)} = e_3/\alpha_1$ (remember that $\alpha_1 = d_1$). Thus, $b_2^{(1)} = 0$, and $e_3^{(1)} = 0$; $d_2^{(1)} = d_2 - a_1\,m_2^{(1)} = d_2 - \gamma_1\,b_2 = \alpha_2$, and $a_2^{(1)} = a_2 - c_1\,m_2^{(1)} = a_2 - \epsilon_1\,b_2 = \gamma_2\,\alpha_2$; $b_3^{(1)} = b_3 - a_1\,m_3^{(1)} = b_3 - \gamma_1\,e_3 = \beta_3$, and $d_3^{(1)} = d_3 - c_1\,m_3^{(1)} = d_3 - \epsilon_1\,e_3$.

Suppose that the matrix obtained at the end of j-th stage of the Gaussian Elimination Method applied to A contains the elements as presented in this theorem. Also, suppose that $\alpha_{j+2} \neq 0$. Thus, in stage $j+1$, the multipliers are represented by $m_{j+2}^{(j+1)} = b_{j+2}^{(j)}/d_{j+1}^{(j)} = \beta_{j+2}/\alpha_{j+1}$, and $m_{j+3}^{(j+1)} = e_{j+3}^{(j)}/d_{j+1}^{(j)} = e_{j+3}/\alpha_{j+1}$. In this way, $b_{j+2}^{(j+1)} = 0$, and $e_{j+3}^{(j+1)} = 0$;

$$d_{j+2}^{(j+1)} = d_{j+2}^{(j)} - a_{j+1}^{(j)}\,m_{j+2}^{(j+1)} = d_{j+2} - \epsilon_j\,e_{j+2} - \gamma_{j+1}\,\beta_{j+2} = \alpha_{j+2}, \text{ and}$$

$$a_{j+2}^{(j+1)} = a_{j+2}^{(j)} - c_{j+1}^{(j)}\,m_{j+2}^{(j+1)} = a_{j+2} - \epsilon_{j+1}\,\beta_{j+2} = \gamma_{j+2}\,\alpha_{j+2};$$

$$b_{j+3}^{(j+1)} = b_{j+3}^{(j)} - a_{j+1}^{(j)}\,m_{j+3}^{(j+1)} = b_{j+3} - \gamma_{j+1}\,e_3 = \beta_{j+3}, \text{ and}$$

$$d_{j+3}^{(j+1)} = d_{j+3} - c_{j+1}\,m_{j+3}^{(j+1)} = d_{j+3} - \epsilon_{j+1}\,e_{j+3}.$$

Therefore, by mathematical induction, the lemma is demonstrated. □

The next Lemma 4.4 and Theorem 4.5 will guarantee that if a pentadiagonal diagonally dominant matrix has a non null determinant, then Crout's decomposition is always possible.

Lemma 4.4. *Let A be a pentadiagonal diagonally dominant matrix as shown in* (1)*. Consider* (3)*,* (4)*,* (5)*,* (6) *and suppose there is a positive integer k, $1 \leq k < n$, such that $\alpha_i \neq 0, \forall i,\ 0 \leq i \leq k-1$, and $\alpha_k = 0$. Then $\det(A) = 0$.*

Proof. *Firstly, considering the lemma's hypotheses, note that $\alpha_i \neq 0, \forall i,\ 1 \leq i \leq k-1 \longrightarrow |\gamma_i| + |\epsilon_i| \leq 1,\ \forall i,\ 1 \leq i \leq k-1$ (the proof of this result is based on mathematical induction and Lemma 2.2, item I). Now, observe that if $\alpha_1 = 0$, then $d_1 = \alpha_1 = 0$. Since the matrix is diagonally dominant, it follows that $a_1 = c_1 = 0$. Therefore, the first row of matrix A would be null and $\det(A) = 0$.*

If $k = 2$, then $M_1 = d_1 = \alpha_1 \neq 0$ and, according to Theorem 4.1, $\frac{M_2}{\alpha_1} = \alpha_2$. If $\alpha_2 = 0$, then $M_2 = 0$, and $d_1 d_2 - a_1 b_2 = 0$. Note that $\alpha_2 = d_2 - \gamma_1\, b_2$ and $|d_2| \geq |b_2| + |a_2| + |c_2|$. In this way, if $d_2 = 0$, then $|b_2| = |a_2| = |c_2| = 0$. Hence, $\det(A) = 0$. Otherwise, if $d_2 \neq 0$ and $0 = |\alpha_2| = |d_2 - \gamma_1\, b_2| \geq |d_2| - |b_2|$, then $|a_2| = |c_2| = 0$ and $|d_2| = |b_2|$. In this way, $|d_1| = |a_1|$ e $|c_1| = 0$, because $d_1 = (a_1\, b_2)/d_2$, and $|d_1| \geq |a_1| + |c_1|$. Hence, the first row of matrix A is represented by the vector $a_1\,(b_2/d_2, 1, 0, \cdots, 0)$ and the second row is represented by the vector $d_2\,(b_2/d_2, 1, 0, \cdots, 0)$. Thus, $\det(A) = a_1\, d_2\, \det(\tilde{A}) = 0$, because the first two rows of $\tilde{A}$ would be the same.

Suppose that $\alpha_i \neq 0$, $\forall i$, $1 \leq i \leq k-1$ and $\alpha_k = 0$, for $k \geq 3$. If $d_k = 0$, then A will have a null row, consequently, $\det(A) = 0$. Consider $d_k \neq 0$ and $|d_k| \geq |e_k| + |b_k| + |a_k| + |c_k|$. In this way, according to Theorem 4.1, $M_k = 0 \iff \alpha_k = 0$. Since $\alpha_k = d_k - \gamma_{k-1}\beta_k - \epsilon_{k-2} e_k$, and according to the calculations presented in Theorem 3.1 from Almeida and Remigio (2023), it follows that $0 = |\alpha_k| \geq |d_k| - |b_k| - |e_k| \longrightarrow |d_k| \leq |b_k| + |e_k|$. Thus, $a_k = c_k = 0$ and $|d_k| = |e_k| + |b_k|$. Respecting these conditions, if $c_{k-1} = 0$, then $\det(A) = 0$. Indeed, just observe that the elements from matrix A satisfy: $A_{i,j} = 0$, for $j > k$ and $1 \leq i \leq k$. Since $M_k = 0$, the process of row reduction of the matrix corresponding to the principal minor M_k will result in a matrix whose the k-th row is a zero row. Therefore, the row-reduced matrix must also contain the same zero row. Similarly, it is possible to prove that $M_j = 0$, $\forall j, j > k$.

Referring to the previous case, if $c_{k-1} \neq 0$, then $|d_{k-1}| \geq |e_{k-1}| + |b_{k-1}| + |a_{k-1}| + |c_{k-1}| > |e_{k-1}| + |b_{k-1}| + |a_{k-1}|$. Thus, according to Lemma 4.2 and Lemma 2.2 (item II), $|\gamma_{k-1}| = |\overline{\gamma}_{k-1}| < 1$, and $b_k = 0$. In fact, if $b_k \neq 0$, then $\alpha_k \neq 0$, according to the calculations presented in Theorem 3.1. Besides, if $e_k = 0$, then $M_k = d_k\, M_{k-1} \neq 0$. Hence, $e_k \neq 0$. In these conditions, to guarantee $M_k = 0$, we need to have $\gamma_{k-2} = 0$ and $|\epsilon_{k-2}| = 1$, because $\gamma_{k-2} \neq 0$ and $|\gamma_{k-1}| < 1 \longrightarrow |\gamma_{k-1}\, \gamma_{k-2}| + |\epsilon_{k-2}| < |\gamma_{k-2}| + |\epsilon_{k-2}| \leq 1$. In this way, $|\alpha_k| > 0$, because $e_k \neq 0$, and $|\alpha_k| =$

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$|d_k - \beta_k\gamma_{k-1} - \epsilon_{k-2}e_k| \geq |d_k| - |\gamma_{k-1}||\beta_k| - |\epsilon_{k-2}||e_k| \geq |d_k| - |\gamma_{k-1}|(|b_k| + |\gamma_{k-2}||e_k|) - |\epsilon_{k-2}||e_k| \geq |d_k| - |b_k| - (|\gamma_{k-1}|\,|\gamma_{k-2}| + |\epsilon_{k-2}|)|e_k| > |d_k| - |b_k| - |e_k| = 0.$

Now, accordingly to the previous paragraph, consider $|\epsilon_{k-2}| = 1$ *and observe that: (1) if* $k = 3$*, then* $|c_1| = |\alpha_1| = |d_1| \geq |a_1| + |c_1|$*; hence,* $a_1 = 0$*. (2) If* $k = 4$*, then* $|c_2| = |\alpha_2| = |d_2 - b_2\gamma_1| \geq |d_2| - |b_2| \geq |a_2| + |c_2|$*; hence,* $a_2 = 0$*. (3) If* $k > 4$*, then* $|c_{k-2}| = |\alpha_{k-2}| = |d_{k-2} - \beta_{k-2}\gamma_{k-3} - \epsilon_{k-4}e_{k-2}| \geq |d_{k-2}| - |b_{k-2}| - |e_{k-2}| \geq |a_{k-2}| + |c_{k-2}|$*; hence,* $a_{k-2} = 0$*. The next step of the proof is to use Lemma 4.3.*

In Lemma 4.3, the superscript $j = 0$ *means that we are considering the original elements from matrix* A*, before applying the Gaussian Elimination Method. In this way, at the end of the* j*-th stage,* $j = k-3$*, we can conclude that the rows* $k-2$ *and* k *will be represented, respectively, by the following vectors:* $(0, \cdots, 0,\, \alpha_{k-2},\, \gamma_{k-2}\,\alpha_{k-2},\, c_{k-2},\, 0, 0, 0, \cdots, 0)$*, and* $(0, \cdots, 0,\, e_k,\, b_k,\, d_k,\, a_k,\, c_k, 0, \cdots, 0)$*, if* $k > 3$*. The number of zeros at the beginning of each one of these vectors is equal to* $k - 3$*. There is no zero at the beginning of the previous vectors, if* $k = 3$*. Besides, the number of zeros after* c_i *is equal to* $n - i - 2$*; if* $i = n - 2$*, then there is no zero after* c_i*. Remembering that* $\gamma_{k-2},\, b_k,\, a_k,\, c_k$ *are all equal to zero,* $|d_k| = |e_k|$*, and* $|\alpha_{k-2}| = |c_{k-2}|$*, then the previous vectors are given by* $\alpha_{k-2}\,(0, \cdots, 0,\, 1,\, 0,\, \epsilon_{k-2},\, 0, 0, 0, \cdots, 0)$*, and* $e_k\,(0, \cdots, 0,\, 1,\, 0,\, \frac{d_k}{e_k},\, 0, 0, 0, \cdots, 0)$*. Next, we are going to prove that* $\frac{d_k}{e_k} = \epsilon_{k-2}$*. Therefore,* $\det(A) = 0$*.*

Note that the elementary operation we are using in Gaussian Elimination does not change neither the determinant of A*, nor its principal minors. In this way, we are going to use the notation* $M_k^{(k-2)}$ *to refer to the determinant of the matrix that is resulting from the removal of the row* $k - 2$ *and column* k *from the matrix associated with the end of the* j*-th stage,* $j = k - 3$*, of the Gaussian Elimination applied to the matrix corresponding to the principal minor* M_k*. Similarly, the following notation is used* $M_k^{(k-1)}$*. Besides, by Lemma 4.3, the elements* a_{k-1} *and* c_{k-2} *are not changed by the Gaussian Elimination at the end of the stage* $j = k - 3$*.*

Using Laplace Expansion by the last column of the matrix associated with the end of the j*-th stage,* $j = k - 3$*, of the Gaussian Elimination applied to the matrix corresponding to the principal minor* M_k*, we obtain that* $0 = M_k = d_k\,M_{k-1} - a_{k-1}\,M_k^{(k-1)} + c_{k-2}\,M_k^{(k-2)}$*. Now, observe that* $M_k^{(k-1)} = 0$*, because the rows* $k - 2$ *and* $k - 1$ *of the matrix associated with this determinant are represented, respectively, by the vectors* $\alpha_{k-2}\,(0, \cdots, 0,\, 1,\, 0)$ *and* $e_k\,(0, \cdots, 0,\, 1,\, 0)$ *(remember that* $b_k = 0$*, in this concerned case). The last two rows corresponding to the determinant* $M_k^{(k-2)}$ *are represented, respectively, by the following vectors:* $(0, \cdots, 0,\, \beta_{k-1},\, d_{k-1}^{(j)})$ *and* $(0, \cdots, 0,\, e_k,\, 0) =$

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$\frac{e_k}{\alpha_{k-2}}(0, \cdots, 0, \alpha_{k-2}, 0)$. *Switching these two rows and remembering that* $\gamma_{k-2} = 0$*, we obtain that* $M_k^{(k-2)} = -\frac{e_k}{\alpha_{k-2}} M_{k-1}$. *In this way,* $0 = M_k = d_k M_{k-1} - c_{k-2} \frac{e_k}{\alpha_{k-2}} M_{k-1} = (d_k - \epsilon_{k-2} e_k) M_{k-1} \longrightarrow d_k - \epsilon_{k-2} e_k = 0 \longrightarrow \frac{d_k}{e_k} = \epsilon_{k-2}$. *Therefore,* $\det(A)$ *= 0.* □

Theorem 4.5. *Let* A *be a pentadiagonal diagonally dominant matrix as shown in* (1)*. If* $\det(A) \neq 0$*, then the parameters* (3) *of Crout's Decomposition are non-zero,* $\alpha_i \neq 0, \forall i, 1 \leq i \leq n$*. Therefore,* A *has an* LU *decomposition and* $A = LU$*.*

Proof. *Suppose there is* j*,* $1 \leq j \leq n$*, such that* $\alpha_j = 0$*. If* $j = 1$*, then* $0 = \alpha_1 = d_1$*. Thus,* $\det(A) = 0$*, by Lemma 4.4. If* $j > 1$*, then it would exist a positive integer* k *(well-ordering principle) such that* $\alpha_i \neq 0, \forall i, 0 \leq i \leq k-1$*, and* $\alpha_k = 0$*. Thus,* $\det(A) = 0$ *(by Lemma 4.4). Therefore, if there was* j*,* $1 \leq j \leq n$*, such that* $\alpha_j = 0$*, then* $\det(A) = 0$*, and we would obtain a contradiction.* □

5 Reverse-permuted matrix

In this section, we will present another technique for detecting in a simple way when a diagonally dominant pentadiagonal matrix A is non-singular and has an LU decomposition. This technique takes in consideration the next Definition 5.1 on reverse-permuted matrix of a matrix A (denoted by $\mathcal{A}$).

Firstly, we define a reverse-permuted matrix as follows.

Definition 5.1. *The reverse-permutated matrix of an order* n *matrix* A *will be denoted by* $\mathcal{A}$*. This matrix is obtained from* A *in two steps. First, we need to carry out a sequence of row permutations in matrix* A*. Next, we need to carry out a sequence of column permutations in this last permuted matrix. These two steps are described as follows: (1º) each row* i *from* A *must be permuted with the row* $n-i+1$*; (2º) each column* j *from the permuted matrix obtained in the first step must be replaced with column* $n-j+1$*.*

In the next example, we observe that the elementary operations performed on rows and columns of matrix A will give rise to a new matrix which considers the elements of the matrix A rewritten from bottom to top and from right to left.

Example 6 (Pentadiagonal reverse-permuted matrix). Consider the matrices below.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \text{ and } \mathcal{A} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The matrix A does not belong to the set P_D because the third row of A does not satisfy none of the six items ((a), (b), (c), (d), (e), and , (f)) described in Definition 2.1. However, rows 1 to 6 from the reverse-permuted matrix $\mathcal{A}$ satisfy the conditions presented in Definition 2.1: rows $1, 2,$ and 4 - item (a); rows 3 and 5 - item (c); row 6 - item (f). Therefore, by Theorem 3.1 from Almeida and Remigio (2023), the matrix $\mathcal{A}$ belongs to the set P_D.

The elementary operation which swaps two rows (or two columns) of a matrix can be represented by a permutation matrix. To swap two rows l and m from a matrix A, we just multiply the permutation matrix $P = P^{(l,m)}$ by A. This matrix permutation is obtained from the identity matrix I by permuting the rows l and m. Besides, if we consider AP, then the columns l and m from A will be permuted.

According to previous notation, it is evident that every permutation matrix is invertible, because $P\,P = I$. Therefore $P^{-1} = P$.

The commutative property $P^{(l,m)}\,P^{(r,s)} = P^{(r,s)}\,P^{(l,m)}$ is valid if l, m, r, s are distinguished two by two (see Proposition 5.2). To prove this property, we consider the following notation:

- the elements of identity matrix I will be denoted by $\delta_{ij} = \begin{cases} 1, & if \;\; i = j, \\ 0, & if \;\; i \neq j; \end{cases}$
- the elements of matrix $P^{(l,m)}$ will be denoted by ν_{ij};
- the elements of matrix $P^{(r,s)}$ will be denoted by ρ_{ij}.

Proposition 5.2. *Given the matrices of order n, $C = P^{(l,m)}\,P^{(r,s)}$ and $F = P^{(r,s)}\,P^{(l,m)}$, we can prove that $C = F$, if l, m, r, s are distinguished two by two.*

Proof. *Using standard procedure, it will be shown that each element c_{ij} from matrix C is equal to the respective element f_{ij} from matrix F.*

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Suppose that i and j are distinct from l, m, r, s. Thus

$$c_{ij} = \sum_{k=1}^{n} \nu_{ik}\, \rho_{kj} = \sum_{k=1}^{n} \delta_{ik}\, \rho_{kj} = \delta_{ii}\, \rho_{ij} = \delta_{ij} = \begin{cases} 1, & if \;\; i = j, \\ 0, & if \;\; i \neq j. \end{cases}$$

The other cases are (I) $i = l$; (II) $i = m$; (III) $i = r$; (IV) $i = s$. We will show only the first case (I), the other cases follow similarly.

$$\text{(I) } i = l \longrightarrow c_{lj} = \sum_{k=1}^{n} \nu_{lk}\, \rho_{kj} = \sum_{k=1}^{n} \delta_{mk}\, \rho_{kj} = \delta_{mm}\, \rho_{mj} = \delta_{mj} = \begin{cases} 1, & if \;\; j = m, \\ 0, & if \;\; j \neq m. \end{cases}$$

The elements from matrix F are calculated as follows. First, consider that i and j are distinct from l, m, r, s. Then

$$f_{ij} = \sum_{k=1}^{n} \rho_{ik}\, \nu_{kj} = \sum_{k=1}^{n} \delta_{ik}\, \nu_{kj} = \delta_{ii}\, \nu_{ij} = \delta_{ij} = \begin{cases} 1, & if \;\; i = j, \\ 0, & if \;\; i \neq j. \end{cases}$$

The other cases are (I) $i = l$; (II) $i = m$; (III) $i = r$; (IV) $i = s$. We will show only the first case (I), the other cases follow similarly.

$$\text{(I) } i = l \longrightarrow f_{lj} = \sum_{k=1}^{n} \rho_{lk}\, \nu_{kj} = \sum_{k=1}^{n} \delta_{lk}\, \nu_{kj} = \delta_{ll}\, \nu_{lj} = \nu_{lj} = \delta_{mj} = \begin{cases} 1, & if \;\; j = m, \\ 0, & if \;\; j \neq m. \end{cases}$$

According to the previous items, $C = F$. Therefore, $P^{(l,m)}\, P^{(r,s)} = P^{(r,s)}\, P^{(l,m)}$. □

Remark: Using mathematical induction, it can be shown that the order of factors does not change the product in a finite multiplication of permutation matrices with the same characteristics of the matrices presented in Proposition 5.2. Therefore, if $P = P_1\, P_2\, \cdots\, P_t$ and $\sigma : \{1, 2, \cdots, t\} \longrightarrow \{1, 2, \cdots, t\}$ is a bijective function, then $P_{\sigma(1)}\, P_{\sigma(2)}\, \cdots\, P_{\sigma(t)} = P$.

Remark: Let $P = P_1\, P_2\, \cdots\, P_t$ be a product of permutation matrices, with the same characteristics as the matrices presented in Proposition 5.2. Then, $P^{-1} = P_t\, P_{t-1}\, \cdots\, P_2\, P_1 = P$.

It will be shown in the next proposition that $\det(\mathcal{A}) = \det(A)$.

Proposition 5.3. *Let $\mathcal{A}$ be the reverse-permuted matrix of an order n matrix A. Then, $\det(\mathcal{A}) = \det(A)$.*

Proof. *Denote by P_k the elementary matrix corresponding to the elementary operation which permutes row k with row $n - k + 1$ from matrix A. If n is an even number, then $t = n/2$ permutation matrices will be used to obtain the reverse-permuted matrix of a matrix A. If n is and odd number, then $t = (n - 1)/2$ permutation matrices will be used to obtain the reverse-permuted matrix and, in* *this case, the row* $k = (n+1)/2$ *will not be altered. In this way,* $\mathcal{A} = P_t\, P_{t-1}\, \cdots\, P_1\, A\, P_1\, \cdots\, P_{t-1}\, P_t$. *Since* $P_k\, P_k = I$*, it follows that* $\det(P_k)^2 = \det(P_k\, P_k) = 1$. *Therefore,* $\det(\mathcal{A}) = \det(A)$. $\square$

The Proposition 5.4 states that if A is a pentadiagonal diagonally dominant matrix, then $\mathcal{A}$ is also a pentadiagonal diagonally dominant matrix.

Proposition 5.4. *The reverse-permuted matrix of a pentadiagonal diagonally dominant matrix is also a pentadiagonal diagonally dominant matrix.*

Proof. *Since the matrix* $A = (A_{ij})$ *of order* n *is pentadiagonal, then* $A_{ij} = 0$*, whenever* $|i-j| > 2$. *Let* $\mathcal{A} = P_t\, P_{t-1}\, \cdots\, P_1\, A\, P_1\, \cdots\, P_{t-1}\, P_t$ *be the reverse-permuted matrix of matrix* A. *The elements of matrix* $\mathcal{A}$ *are denoted by* $\mathcal{A}_{ij}$. *According to the remarks given after Proposition 5.2 and considering* $P = P_1\, \cdots\, P_{t-1}\, P_t$*, we obtain that* $\mathcal{A} = P\, A\, P$. *Therefore, considering an element* $\mathcal{A}_{ij}$ *from matrix* $\mathcal{A}$*, where* $|i-j| > 2$*, it follows that* $\mathcal{A}_{ij} = A_{n-i+1,n-j+1} = 0$*, since* $|i-j| > 2 \iff |(n-i+1)-(n-j+1)| > 2$. *Moreover,* $|\mathcal{A}_{ii}| \geq |\mathcal{A}_{i,i-2}| + |\mathcal{A}_{i,i-1}| + |\mathcal{A}_{i,i+1}| + |\mathcal{A}_{i,i+2}| \iff |A_{n-i+1,n-i+1}| \geq |A_{n-i+1,n-i+3}| + |A_{n-i+1,n-i+2}| + |A_{n-i+1,n-i}| + |A_{n-i+1,n-i-1}|$. $\square$

We next show in Theorem 5.5 that if $\mathcal{A}$ has a Crout's decomposition, and $\det(\mathcal{A}) \neq 0$, then A has a Crout's decomposition and $\det(A) \neq 0$.

Theorem 5.5. *Let* A *be a pentadiagonal diagonally dominant matrix* (1) *and let* $\mathcal{A}$ *be the reverse-permuted matrix of* A. *If* $\mathcal{A}$ *has a Crout's decomposition, and* $\det(\mathcal{A}) \neq 0$*, then* A *has a Crout's decomposition and* $\det(A) \neq 0$.

Proof. *Let* $\mathcal{A}$ *be the reverse-permuted matrix of matrix* A*, where* A *is a pentadiagonal diagonally dominant matrix. According to Proposition 5.4, we know that* $\mathcal{A}$ *is a pentadiagonal diagonally dominant matrix. Besides,* $\det(\mathcal{A}) \neq 0$ *and, according to Proposition 5.3,* $\det(A) = \det(\mathcal{A})$. *Therefore, by Theorem 4.5, the matrix* A *has a Crout's decomposition.* $\square$

We finish this section by showing an application of the previous results. To do this, we return to **Example 6** to observe that $\det(\mathcal{A}) \neq 0 \longrightarrow \det(A) \neq 0$ (see Proposition 5.3). Besides, if $\mathcal{A}$ belongs to the set P_D (Definition 2.1), then $\mathcal{A}$ has a Crout's decomposition, and $\det(\mathcal{A}) \neq 0$. Therefore, by Theorem 5.5, A has Crout's decomposition and $\det(A) \neq 0$.

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6 Conclusions

We present in this work new criteria (sufficient conditions) to identify non-singular pentadiagonal (or tridiagonal) matrices that admit an LU decomposition. These criteria are simple, easy to implement, and they consider non strictly diagonally dominant matrices. The conditions are simple because they can be verified simultaneously and with low operational cost and do not require the computationally expensive calculations of recursive processes like the Crout's method and methods based on determinants, as we can see in Definition 2.1, Theorem 3.1, and Theorem 3.2. Accordingly Lemma 4.4, and Theorem 4.5, that were proved in Section 4, if A is a pentadiagonal diagonally dominant matrix and $\det(A) \neq 0$, then A has an LU decomposition ($A = LU$). Therefore, our objective has been to determine when non strictly diagonally dominant pentadiagonal matrices A, or A^T, or $\mathcal{A}$ (reverse-permuted matrix) belong to the set P_D, or to determine when A, or A^T, or $\mathcal{A}$ is non-singular, by means of Theorems 3.1, and 3.2.

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