Chebyshev polynomials to Volterra-Fredholm integral equations of the first kind

Polinômios de Chebyshev para equações integrais de Volterra-Fredholm do primeiro tipo

Polinomios de Chebyshev para ecuaciones integrales de Volterra-Fredholm de primer tipo

Mohamed Nasseh Nadir[1]
University of Carthage, Faculty of Sciences of Bizerte, Department of Mathematics, Jarzouna, Tunisia
https://orcid.org/0009-0007-1593-8113
Adel Jawahdou[2]
University of Carthage, Institute of Engineering of Bizerte, Department of Mathematics, Jarzouna, Tunisia
https://orcid.org/0000-0002-3551-1236

Abstract: Numerous methods have been studied and discussed for solving ill-posed Volterra integral equations and ill-posed Fredholm integral equations, but rarely for both simultaneously. In this study, we focus on numerically solving the ill-posed Volterra-Fredholm integral equation of the first kind by replacing it with its perturbed counterpart. We employ Chebyshev polynomials of the first kind to solve the perturbed equation. Our findings suggest that this technical approach is superior to the regularization method of Tikhonov. It is simpler, less cumbersome, and this simplicity is demonstrated through various examples.

Keywords: Chebyshev polynomials; Volterra-Fredholm integral equations; ill-posed problems; perturbed equations.

Resumo: Muitos métodos foram estudados e discutidos para a solução da equação integral de Volterra mal-posta e da equação integral de Fredholm mal-posta, mas não de ambas. Neste trabalho resolvemos numericamente a equação integral mal-posta de Volterra-Fredholm de primeiro tipo, substituída por sua equação perturbada, e resolvemos esta última usando os polinômios de Chebyshev de primeiro tipo, sendo que nessa resolução achamos esse método técnico melhor que a regularização de Tikhonov, mais simples e menos embaraçoso; essa simplicidade é verificada por meio de alguns exemplos.

Palavras-chave: polinômios de Chebyshev; equações integrais de Volterra-Fredholm; problema mal-posto; equações perturbadas.

Resumen: Se han estudiado y discutido muchos métodos para resolver la ecuación integral de Volterra mal puesta y la ecuación integral de Fredholm mal puesta, pero no ambas. En este trabajo resolvemos numéricamente la ecuación integral mal puesta de Volterra-Fredholm del primer tipo, reemplazada por su

[1]**Brief curriculum**: Doctor in Mathematics. **Authorship contribution**: Conceptualization, investigation, writing – original draft, writing – review and editing. **Contact**: nadir.mohamednasseh@yahoo.com.

[2]**Brief curriculum**: Doctor in Mathematics. **Authorship contribution**: Conceptualization, investigation, writing – original draft, writing – review and editing. **Contact**: adeljaw2002@yahoo.com.

NADIR, Mohamed Nasseh; JAWAHDOU, Adel. Chebyshev polynomials to Volterra-Fredholm integral equations of the first kind. **REMAT**: Revista Eletrônica da Matemática, Bento Gonçalves, RS, v. 10, n. 1, p. e3002, February 5, 2024. https://doi.org/10.35819/remat2024v10i1id6699.

ecuación perturbada. Resolvemos esta última usando polinomios de Chebyshev del primer tipo y, en esta resolución, creemos que el método técnico de esta resolución es mejor, más sencillo y menos complicado que la regularización de Tikhonov. Esta simplicidad se verifica a través de algunos ejemplos.

Palabras clave: polinomios de Chebyshev; ecuaciones integrales de Volterra-Fredholm; problemas mal planteados; ecuaciones perturbadas.

Submitted: April 28, 2023.

Approved: October 26, 2023.

1 Introduction

It is known that, if we know the disease of an organ in the human body, we easily know the place of the pain, this is direct problem, whereas if we have pain in one part of the body, we cannot know the affected organ, this is inverse problem, on the other hand if a plane lacks kerosene, it risks a crash, this is direct problem, whereas if a plane crashes, we cannot determine the cause, there are many causes. Determining what causes a plane to crash can often take in-depth research and investigation to understand what happened, this is inverse problem.

A considerable number of inverse problems in physical sciences arising in potential theory, radiation, diffusion and heat transfer. In each case the inverse problem will be modeled in terms of an integral equation of the first kind. The concept of inverse problems is well-posedness. That is to say if it satisfies three properties given by Hadamard [1]

1. Existence of the solution;
2. Uniqueness of the solution;
3. Stability of the solution, in other words, the solution depends continuously on the input data.

A problem is ill-posed if at least one of the three well-posedness conditions does not hold. In other words it may not has any solution at all or if there is a solution, it may not be unique or instable, say small changes in measurement data can result in large differences in the corresponding solutions.

Consider the Volterra-Fredholm integral equation of first kind

$$\int_a^x k_1(x,t)\,\varphi(t)\,dt + \int_a^b k_2(x,t)\,\varphi(t)\,dt = f(x); \quad x,t \in [a,b], \tag{1}$$

where the functions $k_1(x,t)$, $k_2(x,t)$, and f are a known continuous functions such that there exists a unique solution $\varphi \in H = C([a,b])$ of the equation (1).

Many methods have been studied and discussed for the solution of ill-posed Volterra integral equation and ill-posed Fredholm integral equation but not both. In [2, 5], the authors use the numerical solution of Fredholm and Volterra integral equations of the first kind using Legendre wavelets collocation method. In [4], regularization methods for first kind Volterra equations are used. Comparison between Taylor and perturbed method for Fredholm and Volterra integral equation of the first kind are studied in [6, 7, 8]. Application of Chebyshev polynomials to Volterra-Fredholm integral equations of the second kind is adopted in [3].

The equation (1) is often written as an operator equation of the form

$$V\varphi + F\varphi = f, \tag{2}$$

where the compact operators V and F are defined for $\varphi \in H$ by

$$V\varphi = \int_a^x k_1(x,t)\,\varphi(t)\,dt, \tag{3}$$

and

$$F\varphi = \int_a^b k_2(x,t)\,\varphi(t)\,dt. \tag{4}$$

If one of the kernels $k_1(x,t)$ or $k_2(x,t)$ is non-degenerate, then the range $R(V+F)$ is non-closed in H. This means that equation (1) is not well posed in space H in the sense that the solution φ of (1) does not depend continuously on the data f, usually we have only an approximation f_δ instead f such that $\|f - f_\delta\| \leq \delta$ for some $\delta > 0$. Noting that the solution φ_δ of $V\varphi + F\varphi = f_\delta$ is not near from the exact solution φ of the equation (1). Therefore, in order to solve this problem with a perturbed data f_δ we must use some procedures.

1. First regularization of Volterra-Fredholm integral equation

For a smooth functions k_1, k_2 and f, the equation (1) can be converted to a Volterra-Fredholm integral equation of the second kind by differentiating with respect to x

$$k_1(x,x)\varphi(x) + \int_a^x \frac{\partial k_1}{\partial x}(x,t)\,\varphi(t)\,dt + \int_a^b \frac{\partial k_2}{\partial x}(x,t)\,\varphi(t)\,dt = f'(x). \tag{5}$$

NADIR, Mohamed Nasseh; JAWAHDOU, Adel. Chebyshev polynomials to Volterra-Fredholm integral equations of the first kind. **REMAT**: Revista Eletrônica da Matemática, Bento Gonçalves, RS, v. 10, n. 1, p. e3002, February 5, 2024. https://doi.org/10.35819/remat2024v10i1id6699.

If $k(x,x) \neq 0$ for all $t \in [a,b]$, the division of the equation (5) by the factor $k(x,x)$ gives us a Volterra-Fredholm integral equation of the second kind where it is known as a well-posed problem:

$$\varphi(x) + \int_a^x k_3(x,t)\,\varphi(t)\,dt + \int_a^b k_4(x,t)\,\varphi(t)\,dt = g(x), \tag{6}$$

where the functions k_3, k_4 and g are given by

$$k_3(x,t) = \frac{1}{k_1(x,x)}\frac{\partial k_1}{\partial x}(x,t),\ k_4(x,t) = \frac{1}{k_1(x,x)}\frac{\partial k_3}{\partial x}(x,t) \text{ and } g(x) = \frac{f'(x)}{k_1(x,x)}.$$

If $k(x,x) = 0$ for $t \in [a,b]$, then we repeat the process of the differentiation up to obtain $\frac{\partial^q k_1}{\partial x^q}(x,t) \neq 0$ for all $t \in [a,b]$, with $k_1,\ k_2$ and f in $C^{q+1}([a,b]\times[a,b])$ and $C^{q+1}([a,b])$, respectively, where q represents the smallest integer for which derivative k_1 of order q does not vanish. So, we have

$$\frac{\partial^q k_1}{\partial x^q}(x,t)\,\varphi(x) + \int_a^x \frac{\partial^{q+1} k_1}{\partial x^{q+1}}(x,t)\,\varphi(t)\,dt + \int_a^b \frac{\partial^{q+1} k_2}{\partial x^{q+1}}(x,t)\,\varphi(t)\,dt = f^{(q+1)}(x). \tag{7}$$

2. **Second regularization of Volterra-Fredholm integral equation**

If an operator of the form $(\mu I + \lambda A)$ is invertible and has a bounded inverse for all $\mu,\ \lambda \in \mathbb{R}_+^*$, then for the equation $A\varphi = f$ with a compact operator A, we take the perturbed equation $\alpha\varphi_{\alpha\delta} + A\varphi_{\alpha\delta} = f_\delta$ with a small α instead the equation $A\varphi = f$. Noting that this approximation is better than the regularization of Tikhonov it is simpler and less embarrassing.

2 Main result

Theorem

Let $\varphi(x) \in C([a,b])$ be the solution of the equation (1) and the operator $(\mu I + \lambda(V+F))$ be invertible with bounded inverse for all $\mu,\ \lambda \in \mathbb{R}_+^*$, then the solution $\varphi_{\alpha\delta}$ of the perturbed equation

$$\alpha\varphi_{\alpha\delta}(x) + \int_a^x k_1(x,t)\,\varphi_{\alpha\delta}(x)dt + \int_a^b k_2(x,t)\,\varphi_{\alpha\delta}(x)dt = f_\delta(x) \tag{8}$$

converges to the exact solution φ provided

$$\|f_\delta - f\| \leq \delta \tag{9}$$

and $\lim\limits_{\alpha\to 0}\frac{\delta}{\sqrt{\alpha}} = 0.$

NADIR, Mohamed Nasseh; JAWAHDOU, Adel. Chebyshev polynomials to Volterra-Fredholm integral equations of the first kind. **REMAT**: Revista Eletrônica da Matemática, Bento Gonçalves, RS, v. 10, n. 1, p. e3002, February 5, 2024. https://doi.org/10.35819/remat2024v10i1id6699.

Indeed, from the subtraction between the equations (8) and (1) , it comes

$$(\alpha I+V+F)(\varphi-\varphi_{\alpha\delta})=(f_\delta-f)+\alpha\varphi\Rightarrow$$
$$(\varphi_{\alpha\delta}-\varphi)=(\alpha I+V+F)^{-1}(f_\delta-f)+\alpha(\alpha I+V+F)^{-1}\varphi$$

and so

$$\begin{aligned}\|\varphi_{\alpha\delta}-\varphi\| &\leq \left\|(\alpha I+V+F)^{-1}\right\|\|f_\delta-f\|+\alpha\left\|(\alpha I+V+F)^{-1}\right\|\|\varphi\| \\ &\leq \frac{\delta}{\sqrt{\alpha}}M+\sqrt{\alpha}M\|\varphi\|\end{aligned}$$

3 Illustrating examples

Example 1

Consider the linear Volterra-Fredholm integral equation

$$\int_{-1}^{x}\cos(t+x)\varphi(t)dt+\int_{-1}^{1}(x+2t)\varphi(t)dt=f(x),$$

where the function $f(x)=4\sin 1-4\cos 1-\frac{1}{2}\sin x-\frac{1}{4}\cos 3x+\frac{1}{4}\cos(x-2)-\frac{1}{2}x\sin x$ chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x)=\sin x.$$

Applying the first Chebyshev polynomial $T_8(x)$ to approximate the solution $\varphi(x),$ that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations for $N=20$, and the errors for $N=40$ with $\alpha=10^{-8}$ (Table 1).

NADIR, Mohamed Nasseh; JAWAHDOU, Adel. Chebyshev polynomials to Volterra-Fredholm integral equations of the first kind. **REMAT**: Revista Eletrônica da Matemática, Bento Gonçalves, RS, v. 10, n. 1, p. e3002, February 5, 2024. `https://doi.org/10.35819/remat2024v10i1id6699`.

Table 1 – The exact and approximate solutions of Example 1 in some arbitrary points, using the first Chebyshev polynomial $T_n(x)$

Points of x	Exact solution	Approximate solution	Error N=40
-1.0000e+00	-8.4147e-01	-8.4147e-01	9.6308e-08
-7.5000e-01	-6.8163e-01	-6.8163e-01	2.7091e-09
-5.0000e-01	-4.7942e-01	-4.7942e-01	1.4548e-09
0.0000e+00	-3.1058e-10	-3.1058e-10	3.1058e-10
5.0000e-01	4.7942e-01	4.7942e-01	5.3113e-10
7.5000e-01	6.8163e-01	6.8163e-01	3.0029e-09
1.0000e+00	8.4147e-01	8.4147e-01	6.5655e-08

Source: Elaborated by the authors.

Example 2

Consider the linear Volterra-Fredholm integral equation

$$\int_{-1}^{x}(t+x)\varphi(t)dt+\int_{-1}^{1}(x-t)\,\varphi(t)dt=f\left(x\right),$$

where the function $f(x)=\dfrac{9}{20}x^5-\dfrac{1}{4}x-\dfrac{1}{5}$ chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x)=x^3.$$

Applying the first Chebyshev polynomial $T_8(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations for $N=20$, and the errors for $N=40$ with $\alpha=10^{-8}$ (Table 2).

Table 2 – The exact and approximate solutions of Example 2 in some arbitrary points, using the first Chebyshev polynomial $T_n(x)$

Points of x	Exact solution	Approximate solution	Error N=40
-1.0000e+00	-1.0000e+00	1.0000e+00	4.3273e-10
-7.5000e-01	-4.2187e-01	-4.2187e-01	9.1738e-11
-5.0000e-01	-1.2500e-01	-1.2500e-01	1.2860e-10
0.0000e+00	0.0000e+00	-2.1176e-10	2.1176e-10
5.0000e-01	1.2500e-01	1.2500e-01	6.2189e-12
7.5000e-01	4.2187e-01	4.2187e-01	1.0355e-10
1.0000e+00	1.0000e+00	1.0000e+00	3.9773e-10

Source: Elaborated by the authors.

Example 3

Consider the linear Volterra-Fredholm integral equation

$$\int_{-1}^{x}(x-t)\varphi(t)dt + \int_{-1}^{1} x\varphi(t)dt = f\left(x\right),$$

where the function $f(x) = 5e^{-1} - 2e^{x} + 4xe^{-1} + xe^{x}$ chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = xe^{x}.$$

Applying the first Chebyshev polynomial $T_8(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations for $N = 20$, and the errors for $N = 40$ with $\alpha = 10^{-8}$ (Table 3).

NADIR, Mohamed Nasseh; JAWAHDOU, Adel. Chebyshev polynomials to Volterra-Fredholm integral equations of the first kind. **REMAT**: Revista Eletrônica da Matemática, Bento Gonçalves, RS, v. 10, n. 1, p. e3002, February 5, 2024. `https://doi.org/10.35819/remat2024v10i1id6699`.

Table 3 – The exact and approximate solutions of Example 3 in some arbitrary points, using the first Chebyshev polynomial $T_n(x)$

Points of x	Exact solution	Approximate solution	Error N=40
-1.0000e+00	-3.6787e-01	-3.6787e-01	5.2495e-07
-7.5000e-01	-3.5427e-01	-3.5427e-01	3.7426e-08
-5.0000e-01	-3.0326e-01	-3.0326e-01	2.7473e-08
0.0000e+00	0.0000e+00	1.4732e-09	1.4732e-09
5.0000e-01	8.2436e-01	8.2436e-01	3.1741e-08
7.5000e-01	1.5877e+00	1.5877e+00	4.5926e-08
1.0000e+00	2.7182e+00	2.7182e+00	5.9256e-07

Source: Elaborated by the authors.

Example 4

Consider the linear Volterra-Fredholm integral equation

$$\int_{-1}^{x} (xt)\varphi(t)dt + \int_{-1}^{1} \cosh(x+t)\,\varphi(t)dt = f(x),$$

where the function $f(x) = \frac{1}{4}\cosh x(4 + e^{2} - e^{-2}) + x^{2}\sinh x - x(\cosh x - e^{-1})$ chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = \cosh x.$$

Applying the first Chebyshev polynomial $T_8(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations for $N = 20$, and the errors for $N = 40$ with $\alpha = 10^{-8}$ (Table 4).

Table 4 – The exact and approximate solutions of Example 4 in some arbitrary points, using the first Chebyshev polynomial $T_n(x)$

Points of x	Exact solution	Approximate solution	Error N=40
-1.0000e+00	1.5430e+00	1.5430e+00	1.9541e-09
-7.5000e-01	1.2946e+00	1.2946e+00	2.0075e-10
-5.0000e-01	1.1276e+00	1.1276e+00	1.3849e-11
0.0000e+00	1.0000e+00	1.0000e+00	6.8679e-10
5.0000e-01	1.1276e+00	1.1276e+00	8.6020e-12
7.5000e-01	1.2946e+00	1.2946e+00	1.8078e-10
1.0000e+00	1.5430e+00	1.5430e+00	2.2303e-09

Source: Elaborated by the authors.

Example 5

Consider the linear Volterra-Fredholm integral equation

$$\int_{-1}^{x}\left(x^{2}+t\right)\varphi(t)dt+\int_{-1}^{1}e^{(x-t)}\varphi(t)dt=f\left(x\right),$$

where the function $f(x)=x^{2}e-xe^{-x}-e^{-x}-x^{2}e^{-x}+\frac{1}{2}e^{-2}e^{x}\left(e^{4}-1\right)$ chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x)=e^{-x}.$$

Applying the first Chebyshev polynomial $T_8(x)$ to approximate the solution $\varphi(x)$, that is to say $\varphi_N(x)$ solution of the algebraic system of linear equations for $N=20$, and the errors for $N=40$ with $\alpha=10^{-8}$ (Table 5).

Table 5 – The exact and approximate solutions of Example 5 in some arbitrary points, using the first Chebyshev polynomial $T_n(x)$

Points of x	Exact solution	Approximate solution	Error N=40
-1.0000e+00	2.7182e+00	2.7182e+00	1.1619e-07
-7.5000e-01	2.1170e+00	2.1170e+00	2.3229e-09
-5.0000e-01	1.6487e+00	1.6487e+00	7.9228e-10
0.0000e+00	1.0000e+00	1.0000e+00	1.9423e-09
5.0000e-01	6.0653e-01	6.0653e-01	1.2104e-09
7.5000e-01	4.7236e-01	4.7236e-01	3.5912e-09
1.0000e+00	3.6787e-01	3.6787e-01	2.4026e-08

Source: Elaborated by the authors.

4 Conclusion

It is well known that the problem of solving the equation (1) is not normally solvable, in other words the range of the operator is not closed, that is to say, the inverse operator is never a continuous operator from its range to the whole space. The goal of this work is to replace the equation (1) ill-posed Volterra-Fredholm integral equations of the first kind by a perturbed equation using Chebyshev polynomials of the first kind where we convert this the perturbed equation into a system of linear algebraic equations. Our numerical experiments shows the efficiency and the stoutness of the proposed method.

References

[1] HADAMARD, J. **Lectures on Cauchy's problem in linear partial differential equations.** New Haven: Yale University Press, 1923. Available in: https://archive.org/details/lecturesoncauchy00hadauoft. Access at: February 5, 2024.

[2] KUMAR, J.; MANCHANDA, P.; POOJA. Numerical solution of Fredholm integral equations of the first kind using Legendre wavelets collocation method. **International Journal of Pure and Applied Mathematics**. v. 117, n. 1, p. 33-43, 2017. DOI: https://doi.org/10.12732/ijpam.v117i1.4.

[3] LAKHAL, A.; NADIR, M.; NADIR, M. N. Application of Chebyshev polynomials to Volterra-Fredholm integral equations. **Australian Journal of Mathematical Analysis and Applications**. v. 19, n. 2, p. 1-8, 2022. Available in: https://ajmaa.org/searchroot/files/pdf/v19n2/v19i2p8.pdf. Access at: February 5, 2024.

[4] LAMM, P. K. **A Survey of Regularization Methods for First-Kind Volterra Equations**. Vienna, New York: Springer, 2000, p. 53-82. Available in: https://users.math.msu.edu/users/lamm/Preprints/Mt_Holyoke_Survey/index.html. Access at: February 5, 2024.

[5] MALEKNEJAD, K.; KAJANI, M. T.; MAHMOUDI, Y. Numerical solution of linear Fredholm and Volterra integral equations of the second kind using Legendre wavelets. **Journal of Sciences, Islamic Republic of Iran**. v. 13, n. 2, p. 161-166, 2002. Available in: https://journal.ut.ac.ir/article_31744_3af2254cc9e8b974559cf3ec796e9692.pdf. Access at: February 5, 2024.

[6] NADIR, M.; BENDJABRI, N. On the invertibility of the Cauchy singular integral. **International Journal of Mathematics and Computation**. v. 29, n. 2, p. 113-118, 2018. Available in: http://www.ceser.in/ceserp/index.php/ijmc/article/view/5496. Access at: February 5, 2024.

[7] NADIR, M.; DJAIDJA, N. Approximation method for Volterra integral equation of the first kind. **International Journal of Mathematics and Computation**. v. 29, n. 4, p. 67-72, 2018. Available in: http://www.ceser.in/ceserp/index.php/ijmc/article/view/5677. Access at: February 5, 2024.

[8] NADIR, M.; DJAIDJA, N. Comparison between Taylor and perturbed method for Volterra integral equation of the first kind. **Numerical Algebra, Control and Optimization**. v. 11, n. 4, p. 487-493, 2021. DOI: https://doi.org/10.3934/naco.2020039.