

Generalized Graph Product and its Application on Generating Solutions of the Millennium n -Queens Problem

Produto Generalizado de Grafos e suas Aplicações na Geração de Soluções para o Problema Milenar das n -Damas

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Abstract: The generalized Kronecker graph product was introduced by Figueroa-Centeno *et al.* (2008). Later, Baca *et al.* (2018) used it for obtaining solutions of the n -queens problem on larger boards from solutions on smaller boards. In this paper, we generalize the graph product and the recent results by Baca *et al.* (2018), obtaining a larger class of solutions by knowing solutions on lower size boards in advance. We finalize the paper stating a couple of conjectures regarding conditions for obtaining composite solutions via graph product.
Keywords: (Modular) n -Queens Problem; Graphs; Graph Theory; Graph Product.

Resumo: O produto de grafos de Kronecker generalizado foi introduzido por Figueroa-Centeno *et al.* (2008). Posteriormente, Baca *et al.* (2018) usaram-o para obter soluções do problema das n -damas em tabuleiros maiores a partir de soluções de tabuleiros de tamanho inferior. Neste artigo, generalizamos o produto de grafos e os resultados recentes de Baca *et al.* (2018), obtendo uma classe maior de soluções conhecendo

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antecipadamente soluções em tabuleiros de menor tamanho. Finalizamos o artigo apresentando algumas conjecturas a respeito das condições de obtenção de soluções compostas via produto de grafos.

Palavras-chave: Problema (Modular) das n -Damas; Grafos; Teoria de Grafos; Produto de Grafos.

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1 Introduction

The n -queens problem was proposed by Bezzel in 1848 (BEZZEL, 1848) for the 8×8 board consisting on placing 8 queens on it without any queen attacking each other. The queens, in chess, move back and forth in both straight lines (row columns) and diagonals; so in order for two of them do not attack each other, they cannot be on the same column, row, or diagonal of the board.

In 1869, Lionnet (LIONNET, 1869) generalized it for the n -queens problem on a $n \times n$ board. Gauss attempted to solve the original 8×8 problem in circa 1850, managing to find 72 solutions. Later, in 1874, Pauls (PAULS, 1874) found all the 92 solutions.

The problem of finding a single solution for the $n \times n$ board has long been solved and nowadays we have various closed formulae for known solutions, such as Pauls' (PAULS, 1874). For a survey on the subject carrying a large amount of specific solutions, see Bell and Stevens (2009). Nevertheless, the problem of finding or classifying all the solutions for a generic n remains unsolved.

In this paper, we study a way of obtaining solutions of boards of bigger size from solutions of boards of lower size. The idea is to generalize the method of Baca *et al.* (2018), where they used the digraph product \otimes_h introduced by Figueiroa-Centeno *et al.* (2008) to generate solutions of boards of size $mn \times mn$ from solutions of boards with sizes $m \times m$ and $n \times n$. Such product was used by Figueiroa-Centeno *et al.* (2008) and Lopez *et al.* (2012) in the context of graph labelings. Baca's idea was a generalization of a known result by Polya and later Rivin *et al.* (See Polya (1918), Rivin, Vardi and Zimmerman (1994)). However, in Baca *et al.* (2018) the solution of an mn -sized board is a block combination of one m -sized solution and multiple n -sized solutions. Here we extend the result, showing that it is possible to obtain an even bigger class of solutions by non-uniform sized block compositions of solutions of smaller boards, given in advance.

The paper is organized as follows: In Section 2, we introduce the generalized product. In Section 3, we generalize Theorem 2.1 of Baca *et al.* (2018), which shows the possibility of constructing solutions of larger boards from solutions of smaller boards using the generalized product of the previous section and provide some examples. In Section 4, we give some conjectures and remarks about possible future research.

2 Notations and the Generalized Graph Product

In this section, we establish notations and preliminary results which will be used in the rest of the paper and we also generalize the \otimes_h -product introduced in Figueiroa-Centeno *et al.* (2008). In Subsection 2.1, we establish the necessary notations and generalize the product in Subsection 2.2.



2.1 Notations and Results

A graph $G = (V, E)$ is a set of vertices V and a set of edges E , where the elements of E are pairs of elements (v_1, v_2) (abbreviated v_1v_2) of V . When the pairs of elements that compose edges of E are ordered we say that G is a digraph. If $G = (V, E)$ we abbreviate the sentence “the set of vertices of G is V ” as $V = V(G)$ and similarly for the set of edges. The degree of a vertex v of a graph G is the number of vertices v' in V such that $vv' \in E$. If G is a digraph we have the outdegree and the indegree, denoting respectively the number of vertices v' such that vv' and $v'v \in E$. A graph is said regular if the degree of all its vertices is equal to a constant k . Alternatively, we also say that the graph is k -regular. A digraph is k -regular if all its vertices have the same indegree and outdegree equal to k . The number of elements of a set A is denoted $|A|$. Given a graph $G = (V, E)$, if $|V| = n$, we enumerate the vertex set as $V = \{1, \dots, |V|\}$. Additionally, the adjacency matrix of G is an $n \times n$ matrix A such that $A_{ij} = 1$ if the edge $ij \in E$ and $A_{ij} = 0$ otherwise. It is common not to allow loops (edges of the form vv) in the definition of a graph (or digraph), but in this paper we consider such possibility allowed. For a more detailed treatment of graph theory see, for instance, Bollobas (2002).

A vertex labeling of a graph (or digraph) $G = (V, E)$ is a bijective function $l : V \rightarrow \{1, \dots, |V|\}$. In this paper, we consider only vertex labelings of a graph (or digraph) so we will omit the word “vertex” and will call vertex labelings only “labelings”. For more detailed theory on graph labelings see Lopez and Muntaner-Batle (2017). It suffices to say this theory studies which graphs have labelings, or studies counting the labelings, that satisfy conditions (graceful, magic, harmonious, etc.) that are similar to the following: A queen-labeling of a graph (or digraph) is a labeling satisfying:

$$\forall u_1u_2, v_1v_2 \in E, l(u_1) + l(u_2) \neq l(v_1) + l(v_2); \quad l(u_1) - l(u_2) \neq l(v_1) - l(v_2). \quad (1)$$

Since a labeling is bijective, and we do not study multiple labelings for one digraph, we identify a queen labeled digraph with one where for each vertex $v \in V = \{1, \dots, |V|\}; l(v) = v$.

Queen-labelings were introduced in Bloom *et al.* (2011), and in the same paper the following result was proved:

Proposition 2.1. *There is a bijection on the set of solutions of the n -queens problem and the queen-labelled 1-regular digraphs G with $|V(G)| = n$.*

Observe that when a queen-labeled digraph represents a solution to an n -queens problem, the digraph must be 1-regular, its vertices must be indexed by $V = \{1, \dots, n\}$, and its adjacency matrix indexed by V is a permutation matrix whose 1 entries are the placements of the queens.

A variant of the n -queens problem is the modular (or toroidal) n -queens problem, which consists of placing n queens on an $n \times n$ board such that no queen attacks another with the extra convention that the board is toroidal, that is, the first line is the $(n + 1)$ -th line and similarly for the first column. In this problem, it is well known that there exist solutions iff $n \pmod 6 = \pm 1$ (POLYA, 1918), however, as in the classic problem, the number or the characterization of all the solutions when they exist is unknown.

We denote the set of the solutions of the classic problem by Q_n , and the set of the solutions of the modular problem by M_n . A solution $Q \in Q_n$ (resp. M_n) can be also described as a function



$\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that $\pi(i) = j$ if in the placement of Q on the board there is a queen on the row i and on the column j , in other words, $(i, \pi(i))$ is an edge in the queen-labelled 1-regular digraph. Since there has to be exactly a queen on each row/column, each $Q \in Q_n$ (resp. M_n) is clearly a permutation and additionally, the function π which describes Q must satisfy that $\pi + id$ and $\pi - id$ are injective (bijective, in the M_n case) because there cannot be more than one queen in each difference/sum diagonal (exactly one, in the M_n case). For more detailed formulation see Bell and Stevens (2009). The set of permutations of $\{1, \dots, n\}$ is denoted by Π_n , therefore it holds $M_n \subset Q_n \subset \Pi_n$ (note that if $n \bmod 6 \neq \pm 1$ then $M_n = \emptyset$).

2.2 The Generalized Graph Product

In this subsection, we present the \otimes_h digraph product as introduced in Figueroa-Centeno *et al.* (2008) and extend it so that we can develop our main result.

Given a family of digraphs Γ with the same set of vertices V , a digraph D and a function $h : E(D) \rightarrow \Gamma$, the product $D \otimes_h \Gamma$ is the digraph satisfying:

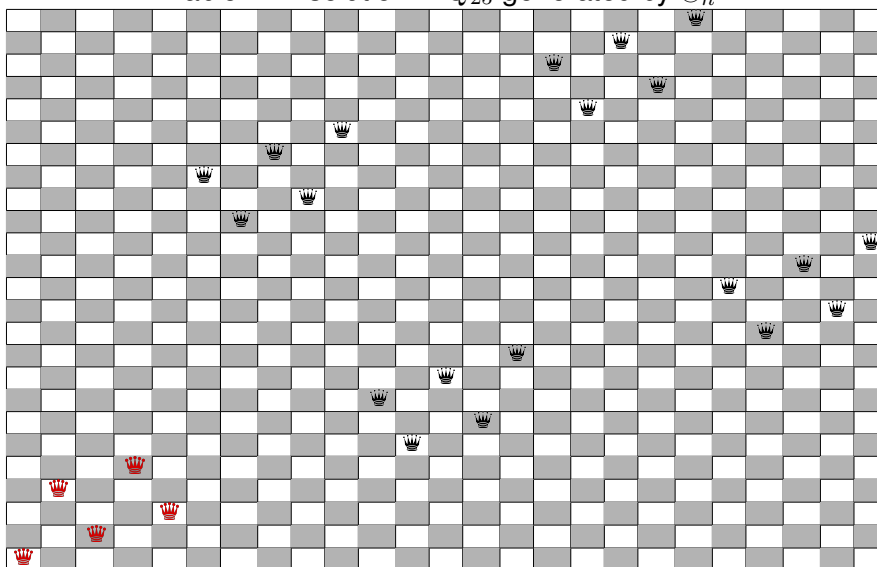
$$V(D \otimes_h \Gamma) = V(D) \times V$$

$$(ai, bj) \in E(D \otimes_h \Gamma) \Leftrightarrow ab \in E(D), \quad ij \in E(h(ab)). \tag{2}$$

In this case, the labelling of $D \otimes_h \Gamma$ is given by $l(a, i) = na + i$.

The graph-product can be visualized by observing the adjacency matrix of $D \otimes_h \Gamma$ (see Table 1): if D has m vertices and each digraph of Γ has n vertices, the matrix of $D \otimes_h \Gamma$ is $mn \times mn$, where seeing it as a block matrix with m^2 blocks of size n , we have null $n \times n$ matrices every time the entry of the adjacency matrix of D is 0 and a non-null matrix when the entry is 1. In this case, the matrix corresponds to an edge ab in D and is exactly the adjacency matrix of the digraph $h(ab) \in \Gamma$.

Table 1: A solution in Q_{25} generated by \otimes_h



Next we formalize a necessary condition for $D \otimes_h \Gamma$ being a solution in Q_{mn} .

Proposition 2.2. *Given a digraph D with m vertices and a family of digraphs $\Gamma = \{\Gamma_1, \dots, \Gamma_m\}$, such that $|\Gamma_i| = n$ for all i , if $D \otimes_h \Gamma \in Q_{mn}$ then $D \in Q_m$ and $\Gamma_i \in Q_n$ for each $\Gamma_i \in \Gamma$.*

Proof.

Since the adjacency matrices of the solutions must have an element equal to 1 on each row and column we have that $D \otimes_h \Gamma \in \Pi_{mn}$, which clearly implies that each $\Gamma_i \in \Pi_n$ and $D \in \Pi_m$. Additionally, it must hold $\Gamma_i \in Q_n$, otherwise there would be two or more elements equal to 1 (corresponding to queens attacking themselves) on the diagonals of the smaller non-null $n \times n$ subboards of the matrix of $D \otimes_h \Gamma$. Suppose that $D \notin Q_m$. Consider the matrix of $D \otimes_h \Gamma$ as a matrix with m^2 blocks of size $n \times n$. Therefore some block-diagonal has two non-null blocks, say a difference-diagonal. Each of these two subboards is an adjacency matrix of some Γ_i . Since each subboard has $2n - 1$ difference-diagonals, we have two boards with n elements equals to 1 (queens) sharing $2n - 1$ diagonals, which implies that some diagonal has two queens, which is a contradiction. ■

Regarding the modular problem, by a similar argument to the one used in the last part of the proof of Proposition 2.2 we see that if $D \otimes_h \Gamma \in M_{mn}$ then $D \in M_m$. But the same idea does not apply for concluding about whether it is true or not that $D \otimes_h \Gamma \in M_{mn}$ implies that $\Gamma_i \in M_n$. To the best of the authors' knowledge it is not known whether such implication holds.

The \otimes_h -product was used in Baca *et al.* (2018) for constructing solutions of boards of size mn from solutions of boards of sizes m and n , like in the example of Table 1: in the example, $\Gamma = \{(53142), (42531)\}$ and D is the permutation (42531). We see that $D \otimes_h \Gamma$ is a solution of Q_{25} .

In this paper, we generalize the results of Baca *et al.* (2018) for compositions of boards of different sizes. However, we cannot use the same definition of \otimes_h because this one demands that the set of vertices of all digraphs of Γ is the same.

Suppose we want to divide the adjacency matrix of a board of size N in blocks, not necessarily of the same size. Each block will have size $i_k \times j_k$, where $\sum_{k=1}^m i_k = \sum_{k=1}^m j_k = N$ and m is the block size of the adjacency matrix. We extend the definition of \otimes_h , and denote it by $\widehat{\otimes}_h$, so that $D \widehat{\otimes}_h \Gamma$ describes such an adjacency matrix in terms of the adjacency matrices of the smaller blocks. The sizes of the smaller blocks can vary. For instance, if D is the permutation (2413) we could have something as in the example of the Table 2.

In Table 2, the blocks corresponding to the edges of D are adjacency matrices of digraphs of Γ of possibly varying sizes. In order for obtaining an actual matrix it is clear that we must have some conditions on the dimensions of the null blocks so that the sum of all the rows and columns equals to N , for instance the block (2, 1) must have dimension 5×4 , since $N = 4 + 4 + 5 + 7 = 20$.

Finally, notice that any solution Q can be seen as a composition $D \otimes_h \Gamma$ where Γ is the family of digraphs of only one vertex and one loop and $D = Q$. This case will be disregarded in our definition because it is not of interest in our context, so unless stated otherwise we refer as block matrices of size greater or equal to 4, since there can not be solutions on boards of sizes 2 and 3. Therefore, we define the generalized \otimes_h -product in the following manner:



Table 2: An adjacency matrix with digraphs in Γ of distinct size

	$[4 \times 4]$		
			$[5 \times 5]$
$[4 \times 4]$			
		$[7 \times 7]$	

Definition 2.3. Let Γ be a family of m digraphs of i_k vertices, $k = 1, \dots, m$, such that $i_k \geq 4$, $\sum_{k=1}^m i_k = N$. Given a 1-regular digraph $D \in \Pi_m$ and a function $h : E(D) \rightarrow \Gamma$, for each edge $ab \in D$ let be $\Gamma_a = h(ab) \in \Gamma$. We define $D \widehat{\otimes}_h \Gamma$ as the digraph satisfying:

$$\begin{aligned} (a, i) \in V(D \widehat{\otimes}_h \Gamma) &\Leftrightarrow a \in V(D), i \in V_{\Gamma_a} \\ ((a, i), (b, j)) \in E(D \widehat{\otimes}_h \Gamma) &\Leftrightarrow ab \in E(D), i, j \in V_{\Gamma_a} \text{ and } ij \in E(\Gamma_a). \end{aligned} \tag{3}$$

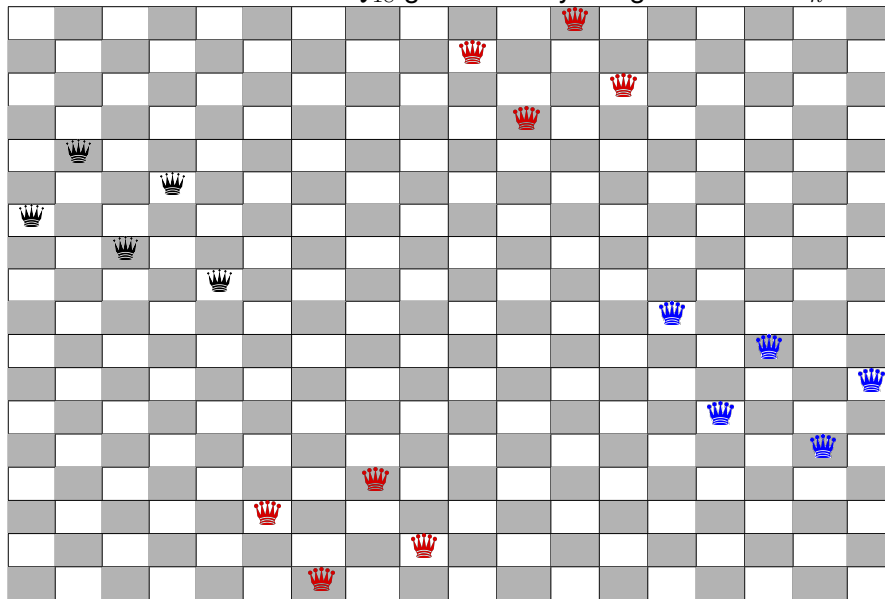
Note that the difference between Definition 2.3 and (2) is that we allow digraphs of Γ with distinct size. Note also that whenever $ab \in E(D)$, there is only one corresponding digraph for such edge, so we are considering Γ_a and Γ_b as the same set, defined by the edge.

We also define the labeling l of $D \widehat{\otimes}_h \Gamma$ such that for every vertex (a, i) it holds $l(a, i) = \sum_{k=1}^{a-1} i_k + i$. Notice that this labeling corresponds to the number of vertex's row on the $N \times N$ board.

The example of Table 3 is a solution in Q_{18} constructed as $D \widehat{\otimes}_h \Gamma$, with $D = (3142) \in \Pi_4$ and $\Gamma = \{(3142), (24135), (13524)\}$. The definition of h is clear by the adjacency matrix.



Table 3: A solution in Q_{18} generated by the generalized \otimes_h



In the case that $D\widehat{\otimes}_h\Gamma$ is a solution in Q_N we get the following result:

Proposition 2.4. Given $D \in \Pi_m$ and a family Γ (possibly repeating) of digraphs Γ_k , each with i_k vertices, $k = 1, \dots, m$ satisfying $i_k \geq 4$ and $\sum_{k=1}^m i_k = N$, if $D\widehat{\otimes}_h\Gamma \in Q_N$ then $\Gamma_k \in Q_{i_k}$.

Proof.

First, notice that each adjacency matrix of Γ_k must be square therefore the blocks corresponding to the edges of D must be squares of size $i_k \times i_k$, and only the null blocks can be rectangular. Since $D\widehat{\otimes}_h\Gamma \in Q_N$ there must be exactly an element equal to 1 on each row and column of its adjacency matrix, hence the blocks corresponding to the matrices of Γ_k are in Π_{i_k} . Additionally, it must hold $\Gamma_k \in Q_{i_k}$ otherwise we would have diagonals with more than one element equal to 1 in the matrix of $D\widehat{\otimes}_h\Gamma$. ■

Observe that unlike in Proposition 2.2, we cannot conclude that $D \in Q_m$. This occurs because since j_k can differ from i_k , the fact that there are two blocks sharing the same block-diagonal on the adjacency matrix of $D \otimes_h \Gamma$ does not imply the blocks share diagonals as in the square composition case, as it is shown in Table 4.



Table 4: Attacking block-diagonals do not imply the diagonals attack each other

5 x 5	5 x 15	5 x 4
4 x 5	4 x 15	4 x 4
15 x 5	15 x 15	15 x 4

In the matrix of Table 4, the blocks 5×5 and 15×4 are on the main block-diagonal, but the squares do not attack each other in the classic sense because there is a lot of space among them, since the blocks of the second block-column have 15 columns.

3 Obtaining Solutions of n -Queens Problem with the Generalized Graph Product

In this section, we use the $\hat{\otimes}_h$ -product, given in the Definition 2.3 for obtaining solutions of the n -queens problem on boards of bigger size from a solution of boards of smaller size.

In Subsection 3.1, we prove a result establishing a sufficient condition for constructing such composite solutions, and in Subsection 3.2 we present some examples.

3.1 Condition for Constructing Solutions with the Generalized Graph Product

We start **with** a result whose proof is based on Theorem 2.1 of Baca *et al.* (2018); in that paper Baca *et al.* label each vertex in $D \otimes_h \Gamma$ as the number of the line it corresponds to in the adjacency matrix (namely the board of size N). Here we extend the idea and obtain a similar result.

As in Baca *et al.* (2018), given two vertices v and v' of a graph (or digraph) G and a labeling l of G , we define $s(v, v') := l(v) + l(v')$ and $d(v, v') := l(v') - l(v)$.

In the following result, we give a formula for computing s and d in our context.



Lemma 3.1. Each edge $((a, b), (i, j)) \in E(D\widehat{\otimes}_h\Gamma)$ satisfies:

$$\begin{aligned}
 s((a, i), (b, j)) &= i + j + \sum_{k=1}^{m(a,b)} 2i_k + \sum_{k=1+m(a,b)}^{M(a,b)} i_k \\
 d((a, i), (b, j)) &= j - i + \left(\pm \sum_{k=a}^{b-1} i_k \right),
 \end{aligned}
 \tag{4}$$

where $m(a, b) = \min\{a - 1, b - 1\}$, $M(a, b) = \max\{a - 1, b - 1\}$ and:

$$\left(\pm \sum_{k=a}^{b-1} i_k \right) = \begin{cases} \sum_{k=a}^{b-1} i_k, & b - 1 \geq a \\ - \sum_{k=b-1}^a i_k, & b - 1 < a \end{cases}.$$

Proof.

Take two vertices (a, i) and (b, j) in $D\widehat{\otimes}_h\Gamma$, recall that the labeling l is given by $l(a, i) = \sum_{k=1}^{a-1} i_k + i$.

Therefore, we have that

$$s((a, i), (b, j)) = l(a, i) + l(b, j) = \sum_{k=1}^{a-1} i_k + i + \sum_{k=1}^{b-1} i_k + j,$$

now by using the definitions of $m(a, b)$ and $M(a, b)$, we see that $s((a, i), (b, j))$ is given by the first equation in (4).

By the other hand, the difference $d((a, i), (b, j))$ is given by

$$d((a, i), (b, j)) = l(b, j) - l(a, i) = j - i + \left(\sum_{k=1}^{m(a,b)} i_k - i_k \right) + \left(\pm \sum_{k=a}^{b-1} i_k \right),$$

proving (4), since $\left(\sum_{k=1}^{m(a,b)} i_k - i_k \right)$ vanishes. ■

Given a set A and an element b , as in Baca *et al.* (2018), we define $A \pm b = \{a \pm b; a \in A\}$ and $s(A)$, $d(A)$ respectively as the sets of the pairwise sums and differences of A . With these notations and with Lemma 3.1 in mind, we are ready to give a sufficient condition for constructing solutions using the $\widehat{\otimes}_h$ -product:

Theorem 3.2. Let $D \in \Pi_m$ and consider a family $\Gamma = \{\Gamma_1, \dots, \Gamma_m\}$ of m solutions $\Gamma_k \in Q_{i_k}$ satisfying $i_k \geq 4$ and $\sum_{k=1}^m i_k = N$. Consider a function $h : E(D) \rightarrow \Gamma$ such that for every two given edges



$ab, \bar{a}\bar{b} \in E(D)$ if the assumptions:

$$\begin{aligned}
 H1) \quad \hat{s} &:= \left| \sum_{k=1}^{m(\bar{a},\bar{b})} 2i_k + \sum_{k=1+m(\bar{a},\bar{b})}^{M(\bar{a},\bar{b})} i_k - \left(\sum_{k=1}^{m(a,b)} 2i_k + \sum_{k=1+m(a,b)}^{M(\bar{a},\bar{b})} i_k \right) \right| < \max\{i_a, i_{\bar{a}}\} \\
 H2) \quad \hat{d} &:= \left| \left(\pm \sum_{k=a}^{b-1} i_k \right) - \left(\pm \sum_{k=\bar{a}}^{\bar{b}-1} i_k \right) \right| < \max\{i_a, i_{\bar{a}}\}
 \end{aligned} \tag{5}$$

imply that:

$$\begin{aligned}
 T1) \quad [s(h(ab)) - \hat{s}] \cap s(h(\bar{a}\bar{b})) &= \emptyset \\
 T2) \quad [d(h(ab)) - \hat{d}] \cap d(h(\bar{a}\bar{b})) &= \emptyset,
 \end{aligned} \tag{6}$$

then $D \widehat{\otimes}_h \Gamma \in Q_N$.

Proof. First, notice that H1) and H2) are equivalent to saying that the sum and difference diagonals of the blocks defined by (a, b) and $(\bar{a}, \bar{b}) \in D$ do not intersect. For instance, two blocks attack each other in their sum diagonal if the difference of their sum diagonals (the number defined as \hat{s} in H1) is less than the greater of the sizes of the blocks (namely, $\max\{i_a, i_{\bar{a}}\}$); this way we see there is necessarily at least one sum-diagonal which passes through both blocks. Conversely, if the condition in H1 is not met then surely the blocks defined by (a, b) and (\bar{a}, \bar{b}) don't intersect.

Since $D \in \Pi_m$ and $\Gamma_{i_k} \in \Pi_n$ for all k , there is exactly one element equal to 1 in each row and column of the adjacency matrix of $D \widehat{\otimes}_h \Gamma$. It suffices to show that there are no two elements equal to 1 in any difference or sum diagonal.

In order for that to happen we need that for every two edges $((a, i), (b, j))$ and $((\bar{a}, \bar{i}), (\bar{b}, \bar{j}))$ in $E(D \widehat{\otimes}_h \Gamma)$, it must hold:

$$s((a, i), (b, j)) \neq s((\bar{a}, \bar{i}), (\bar{b}, \bar{j})), \quad d((a, i), (b, j)) \neq d((\bar{a}, \bar{i}), (\bar{b}, \bar{j})).$$

Using (4), we see that this condition is equivalent to:

$$\begin{aligned}
 i + j + \sum_{k=1}^{m(a,b)} 2i_k + \sum_{k=1+m(a,b)}^{M(a,b)} i_k &\neq \bar{i} + \bar{j} + \sum_{k=1}^{m(\bar{a},\bar{b})} 2i_k + \sum_{k=1+m(\bar{a},\bar{b})}^{M(\bar{a},\bar{b})} i_k, \\
 j - i + \left(\pm \sum_{k=a}^{b-1} i_k \right) &\neq \bar{j} - \bar{i} + \left(\pm \sum_{k=\bar{a}}^{\bar{b}-1} i_k \right).
 \end{aligned} \tag{7}$$

Since (7) must hold for every pair of edges $((a, i), (b, j))$ and $((\bar{a}, \bar{b}), (\bar{i}, \bar{j}))$, this condition must be considered every time the block-diagonals of the adjacency matrix (seen as a block matrix) has intersecting diagonals, that is, every time it holds (H1) or (H2).

By the definitions of \hat{s} and \hat{d} in (5), supposing that every time it holds (H1) and (H2) it implies (T1) and (T2) we see that no diagonals have more than one element equal to 1. ■



Observe that the numbers \hat{s} and \hat{d} of conditions (H1)-(H2) depend only on the digraph D and not on the subboards described by each Γ_k , similarly as what happened in Theorem 2.1 of Baca *et al.* (2018). In fact, they are generalizations of the constant n in that Theorem.

3.2 Examples

In this subsection, we present and discuss some other examples concerning the results of Subsection 3.1.

We begin observing that Theorem 3.2 gives conditions for constructing composite solutions for the classic problem. One could think that with the additional strong hypotheses that D and Γ_k are all also solutions of the respective dimensional modular problem then it holds $D \otimes_h \Gamma \in M_{mn}$, at least for the case that \otimes_h is given as in (2). After all, letting $l(a, i) := n(a - 1) + i$ (number of the line of the corresponding edge on the adjacency matrix), supposing for $ab, \bar{a}\bar{b}$ in D it holds $a \pm b \not\equiv \bar{a} \pm \bar{b} \pmod m$ and in each Γ_k it holds $i \pm j \not\equiv \bar{i} \pm \bar{j} \pmod n$ then we get that $n(a - 1) + i \pm n(\bar{b} - 1) + j \not\equiv n(\bar{a} - 1) + \bar{i} \pm n(\bar{b} - 1) + \bar{j} \pmod n$ (and therefore both sides are also different \pmod{mn}), hence we conclude that no two queens share the same sum/difference diagonal of other.

As a matter of fact, this very idea has been used by Baca *et al.* in the proof of this same result using the same notation (See Theorem 3.1 of Baca *et al.* (2018)).

However, in that paper they included an additional hypothesis ensuring the diagonals don't intersect. A sufficient condition for the argument to hold would be imposing that

$$i \pm j \not\equiv \bar{i} \pm \bar{j} \pmod n, \tag{8}$$

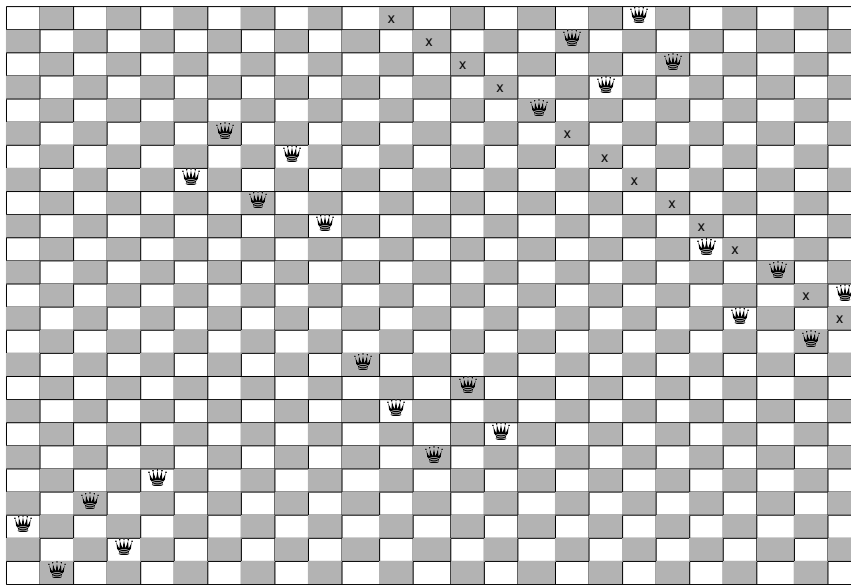
for every edges $ij, \bar{i}\bar{j} \in \Gamma$, given a pair (ai, bj) and $(\bar{a}\bar{i}, \bar{b}\bar{j})$ of edges of $D \otimes_h \Gamma$.

Condition (8) may hold for every digraph Γ_k alone, but when one fixes two distinct pairs $((a, i), (b, j)), ((\bar{a}, \bar{i}), (\bar{b}, \bar{j})) \in D \otimes_h \Gamma$ we cannot relate the integers i and j modulo n because they could belong to different digraphs of Γ . This hasty reasoning may lead one to overlook examples like the one in Table 5, where the product \otimes_h of a solution $D \in M_5$ and a family of solutions in M_5 does not yield a modular solution in M_{25} .

In Table 5, $n = m = 5$. Seeing the queens as quadruples $((a, i), (b, j))$ corresponding to the digraph edges, we have that $a = 1, b = 4, i = 5, j = 1$ and $\bar{a} = 3, \bar{b} = 5, \bar{i} = 2, \bar{j} = 3$. It is true that $a - b \not\equiv \bar{a} - \bar{b} \pmod 5$, however $i - j \pmod 5 = \bar{i} - \bar{j} \pmod 5 = 4$.



Table 5: A composition of modular solutions: two queens do attack each other



Therefore a condition such as (8) is necessary in order to ensure that $D \hat{\otimes}_h \Gamma$ is in fact a modular solution. We formalize this idea in the next result:

Proposition 3.3. *If $D \in M_m$, Γ is a family of solutions of M_n and for every pair of edges $(ai, bj), (\bar{a}\bar{i}, \bar{b}\bar{j}) \in E(D \otimes_h \Gamma)$ condition (8) holds, then $D \otimes_h \Gamma$ is a modular solution of M_{mn} .*

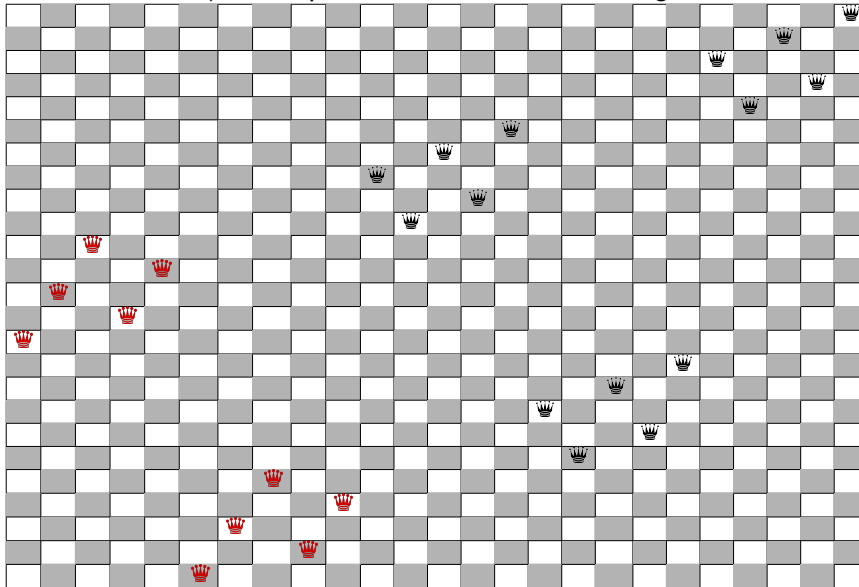
Proof. It follows from (8) and the reasoning above. ■

In Table 6, we have a modular solution of M_{25} given by the product of (53142) and $\Gamma = \{(53142), (35241)\}$. This example was taken from Baca *et al.* (2018).

The relation between \otimes_h and modular solutions can become even less intuitive, as shown in the example in Section 2, given in Table 1. We have a solution of Q_{25} given by the composite product of modular solutions D and a family of modular solutions Γ , however it is itself a solution only of the classic problem, but not of the modular one. Applying on it the translation $t(i, j) = (i, j - 1 \pmod{25})$, we see that the queen (25, 1) gets placed in (25, 25) while the queen (7, 8) is placed in (7, 7), both belonging to the same difference diagonal.

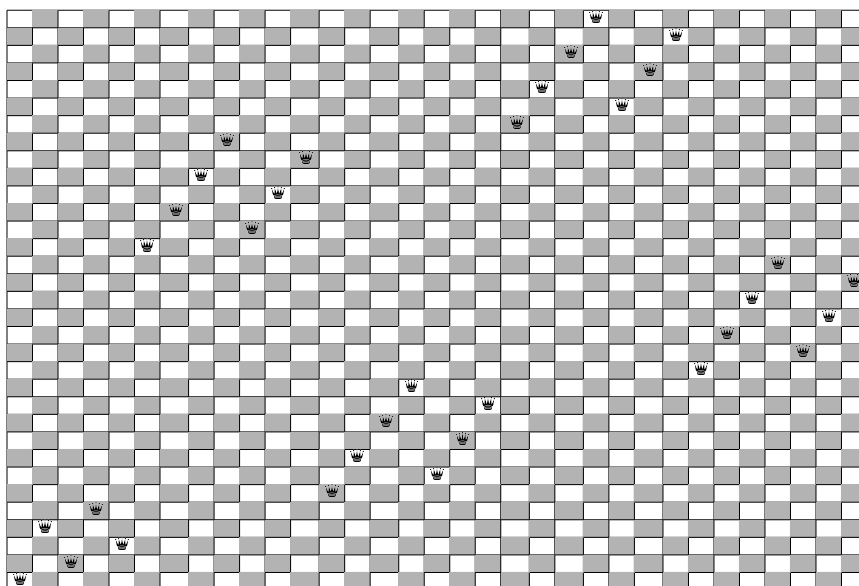


Table 6: (BACA *et al.*, 2018) A composition of solutions of M_5 generates a solution of M_{25}



The generalized product $\hat{\otimes}_h$, given in Definition 2.3 is also capable of generating similar examples (product of modular solutions is only a classic solution), one of them is shown in Table 7: we have a product of some digraphs of order 5 and some of order 7, all modular solutions, which yields a solution in Q_{33} , but not in M_{33} (we can see this because $33 \pmod 6 \neq \pm 1$, therefore, there is no modular solution for such board (POLYA, 1918)).

Table 7: A classic but not modular solution obtained via generalized product



We would like to highlight that instead of using the generalized $\hat{\otimes}_h$ -product we could define the permutation in Π_{mn} assigning each pair (a, i) to $nf(a) + g_k(i)$, where $f \in \Pi_m$ and each $g_k \in \Pi_n$



represent the bigger and smaller solutions, respectively. In Rivin, Vardi and Zimmerman (1994), and Mikhailovski (2018), such notation is used for the particular case where g_k is the same for all k . In those papers, the authors also managed to obtain general results for compositions of the form $nf_j(a) + g(i)$ using only the fact that $f_j \in Q_k$ and g was a modular solutions without any additional hypotheses, however we point out this kind of composition does not translate to the rectangular composition in the sense of this paper: unlike in our case f_j vary and g is fixed instead. Therefore this solutions' adjacency matrix is not of the block-type.

4 Concluding Remarks

In this Section, we finalize the paper with some remarks about the generalized $\widehat{\otimes}_h$ -product and state a couple of conjectures as well as introduce possible future research. The remarks are given in Subsection 4.1 and a brief overall conclusion is given in Subsection 4.2.

4.1 Remarks and Conjectures

We start with a remark about the generalized $\widehat{\otimes}_h$ -product: note that such product, as defined in (2), only yields solutions of boards of composite size, which implies there are no solutions given by a (square) composition of solutions of boards of smaller size in Q_N when N is prime. At first glance this fact has no motive to still hold when we use the generalized $\widehat{\otimes}_h$ -product, since even a prime N can be decomposed on a sum of integers. For instance, $N = 17$ can be written as $4 + 4 + 4 + 5$, therefore splitting conveniently the 17×17 board in blocks of dimensions 4×4 , 4×5 , 5×4 and 5×5 , it should be possible to find a solution given by the generalized product of solutions in Q_4 and Q_5 .

Surprisingly enough, it is not what seems to happen in practice. We found a generalized composite solution in Q_{18} (in Table 3) when $N = 18$, however one can exhaust every possibility of composing solutions in Q_4 and Q_5 for $N = 17$ and check that there is no such kind of solution. Observe that since there are only 2 solutions in Q_4 and 10 in Q_5 , and there are only 4 distinct ways of ordering 3 numbers '4' and a '5', we need to analyze only $4 \cdot 10 \cdot 2^3$ compositions.

Exhausting all the possibilities in a software we found there is no composite solution for $N = 17$ as well as for $N = 19$. However, one can easily (with a software) find examples for $N = 18$ and $N = 20$. Such behavior inspires the conjecture that the impossibility of finding composite solutions with the \otimes_h -product when N is prime extends to the generalized product.

Conjecture 4.1. *Let $D\widehat{\otimes}_h\Gamma$ be a digraph with N vertices given as the product of D and a family Γ such as in Definition 2.3. If $D\widehat{\otimes}_h\Gamma \subset Q_N$ then N is not prime.*

Computing solutions in a software we may stretch the conjecture a little: for $N = 20$ we get various examples of compositions of a 4×4 block matrix, each block with dimension 5×5 , however it is impossible to find a way of obtaining four solutions: two 5×5 , one 4×4 and other 6×6 such that their composition in a 4×4 block matrix 4×4 yields a solution in Q_{20} . This suggests there exists a condition on the form of the decomposition of N in the sum of integers in order for the decomposition



to yield at least one solution given by generalized product of solutions of digraphs with orders given by the numbers of the decomposition.

Conjecture 4.2. Let $D \widehat{\otimes}_h \Gamma$ be a digraph with N vertices given by the product of D and a family $\Gamma = \{\Gamma_1, \dots, \Gamma_m\}$ (possibly equal); each Γ_k with i_k vertices, such that $\sum_{k=1}^m i_k = N$. If $D \widehat{\otimes}_h \Gamma \subset Q_N$ then there is some strong unknown condition on the form of the numbers i_k , $k = 1, \dots, m$.

A topic of future research is addressing Conjecture 4.2 and classifying the ways of decomposing N as a sum of integers such that the existence of a composite solution with such integers as sizes of subboards is granted.

The n -queens problem is hard so the more results on algebraic characterization the better, and as important as characterizing is providing results and conditions for existence as well as establishing limits for the number of solutions on a particular class. With this in mind, another subject for future research is establishing asymptotic lower and/or upper bounds for the number of existing composite solutions of a given N .

4.2 Conclusion

In this paper we have generalized the graph \otimes_h -product for attaining composite solutions of the n -queens problem via adjacency matrix even when the block matrices corresponding to subboards are not of the same size, obtaining a broader class of possible solutions from solutions of smaller boards.

We obtained necessary conditions for constructing such composite solutions for the classical problem in Theorem 3.2 and discussed a previous result by Baca *et al.* (namely Theorem 3.1) about the sufficient condition for obtaining composite solutions (in the sense of 2). We concluded that in such theorem the authors used a hasty argument which does not necessarily hold and must have been added as an additional hypothesis, as stated in Proposition 3.3. We obtained necessary conditions for constructing such composite solutions for the classical problem in Theorem 3.2 and discussed a result generalizing Theorem 3.1 in Baca *et al.* (2018), establishing sufficient conditions for composing a solution in M_{mn} , given in Proposition 3.3.

The possibilities of classifying the solutions of the n queens problem is endless and in this paper we have extended the theory about the class of composite solutions further.

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