


# Numerical study of different methods applied to the one-dimensional transient heat equation

## Estudo numérico de diferentes métodos aplicados à equação transiente do calor unidimensional


Neyva Maria Lopes Romeiro

State University of Londrina (UEL), Department of Math and PGMAC, Londrina, PR, Brazil

 <https://orcid.org/0000-0002-3249-3490>, nromeiro@uel.br


Eduardo Oliveira Belinelli

Federal University of Paraná (UFPR), PPGMNE, Curitiba, PR, Brazil

 <https://orcid.org/0000-0002-5925-086X>, eduardo.belinelli@ufpr.br


Jesika Maganin

Federal University of Paraná (UFPR), PPGMNE, Curitiba, PR, Brazil

 <https://orcid.org/0000-0003-2719-3124>, jesikamaganin@ufpr.br


Paulo Laerte Natti

State University of Londrina (UEL), Department of Math and PGMAC, Londrina, PR, Brazil

 <https://orcid.org/0000-0002-5988-2621>, plnatti@uel.br

Eliandro Rodrigues Cirilo

State University of Londrina (UEL), Department of Math and PGMAC, Londrina, PR, Brazil

 <https://orcid.org/0000-0001-7530-1770>, ercirilo@uel.br

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#### Abstract

This article aims to compare the results obtained by applying three numerical methods: Explicit Euler, Crank-Nicolson, and Multi-stage (R11), in the one-dimensional heat diffusion transient equation with different initial and boundary conditions. The discretization process was performed using the finite difference method. In order to guarantee the convergence of the methods used, consistency and stability were verified by Lax theorem. The results are presented in graphs and tables that contain the data of the analytical solution and the numerical solutions. It was observed that the results obtained by R11 method generated solutions with minor errors.

**Palavras-Chave**

Modelo Matemático  
Método de Diferenças Finitas  
Convergência

**Resumo**

Este artigo tem por objetivo comparar os resultados obtidos pela aplicação de três métodos numéricos: Euler Explícito, Crank-Nicolson e Multi-estágio (R11), na equação transiente da difusão do calor unidimensional com diferentes condições iniciais e de contorno. O processo de discretização foi realizado pelo método de diferenças finitas. Para garantir a convergência dos métodos utilizados foi verificada a consistência e a estabilidade pelo Teorema de Lax. Os resultados são apresentados em gráficos e tabelas que contêm dados da solução analítica e das soluções numéricas. Observou-se que os resultados obtidos pelo método R11 gerou soluções com menores erros.

**1 Introduction**

The transient heat diffusion equation is a linear partial differential equation (PDE) of first-order in time and second-order in space. Transient problems involve the temporal variation of the physical quantities of interest. From the initial and contour values of these quantities, the PDE solution calculates their values in successive time intervals (FORTUNA, 2012).

In this article, we consider the problem of finding numerical solutions for the one-dimensional heat diffusion equation in transient regime with different initial and boundary conditions, described mathematically by

$$u_t - \alpha(x, t)u_{xx} = r(x, t) \quad u(x, t) \in \Omega \times (0, T) \quad (1)$$

$$u(x, 0) = f(x) \quad (2)$$

$$u(x, t) = \phi(x, t) \quad (x, t) \in \partial\Omega \times (0, T), \quad (3)$$

so that  $\alpha(x, t)$  represents thermal diffusivity (BOYCE; DIPRIMA, 1985, FORTUNA, 2012),  $\Omega = [0, L]$  is the spatial domain, and  $(0, T)$  is the temporal domain. The functions  $r(x, t)$ ,  $f(x)$  and  $\phi(x, t)$  describe the source term and the initial and boundary conditions, respectively.

In recent years, several computational techniques were developed to obtain numerical solutions of PDEs. This approach consists of transforming continuous problems into discrete problems, involving both the equation of interest and the geometric domain of solutions. In the latter, the transformation consists of the decomposition of a continuous region into a finite set of points. Then, the

equations of interest must be written in the form of arithmetic operations, depending on the discretized domain.

Several methods in the literature can be used to obtain numerical solutions to PDEs, such as the methods developed in Horváth (2002), Kadalbajoo and Awasthi (2006), Wang, Nakagawa and Yamamoto (2010), Ladeia *et al.* (2013), Gu *et al.* (2018), Hajipour *et al.* (2018), Kazen and Dehghan (2018) and Saita *et al.* (2018). In this article, three methods will be particularly addressed: the Explicit Euler method, Crank-Nicolson (CN), and Multi-stage (R11), which are found in Faragó and Palencia (2002), Araújo and Márquez (2012) and Pereira, Lisboa and Dias Filho (2017). The objective is to verify which method produces numerical solutions closer to analytical solutions. Based on the results obtained, error analysis will be carried out, as well as the time taken to process the solutions. The convergence of the methods is verified through Lax theorem.

The article is accomplished as follows: in section 2, the model discretization process, equation (1), is carried out using the finite difference method. Section 3 presents the convergence theory of numerical methods. In section 4, the convergence analysis (consistency and stability) of the R11 method is verified. Section 5 presents the results obtained in this article. Finally, some considerations are drawn in section 6.

## 2 Numerical model

The geometric domain of solutions, defined in this article, is a region of length  $[0, L]$  in the spatial domain  $x$ , and an interval  $(0, T)$  in the temporal domain  $t$ . Dividing the interval  $(0, L)$  into  $M_x$  equal parts in length  $\Delta x$ , there is  $M_x + 1$  points  $x_i = i\Delta x$ , with  $i = 1, \dots, M_x$ , so that  $\Delta x = L/M_x$ . Similarly, dividing the interval  $(0, T)$  into  $M_t$  equal parts in length  $\Delta t$ , it results in  $M_t + 1$  points  $t_j = j\Delta t$ , with  $j = 1, \dots, M_t$ , so that  $\Delta t = T/M_t$ . Thus, the approximations of the differential terms in equation (1) must be performed according to the points of the discretized geometric domain defined by the set:

$$(x_i, t_j) = (x_0 + i\Delta x, y_0 + j\Delta t) \quad \text{with } i = 1, \dots, M_x \text{ and } j = 1, \dots, M_t, \quad (4)$$

in which  $(x_0, y_0)$  represents the origin of the geometric domain of solutions.

In this article,  $u_{ij}$  is denoted as the analytical solution at point  $(x_i, t_j)$  and for  $U_{ij}$  as the numerical approximate value of  $u_{ij}$ .

The equation (1) is discretized by the finite difference method, given by

$$\frac{U_i^{j+n} - U_i^{k+m}}{\Delta t/s} = \sigma \left( \alpha \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{\Delta x^2} \right) + (1 - \sigma) \left( \alpha \frac{U_{i-1}^j - 2U_i^j + U_{i+1}^j}{\Delta x^2} \right) + r_i^j. \quad (5)$$

In the model discretization process,  $\alpha(x, t)$  was considered constant. In (5), in case that  $\sigma = 0$ ,  $s = 1$ ,  $n = 1$ ,  $k = j$  and  $m = 0$ , the Explicit Euler method is obtained. If  $\sigma = 1/2$ ,  $s = 1$ ,  $n = 1$ ,  $k = j$  and  $m = 0$ , the Crank-Nicolson method is obtained. If  $\sigma = 0$ ,  $s = 2$ ,  $n = 1/2$ ,  $k = j$  and  $m = 0$ , or if  $\sigma = 1$ ,  $s = 2$ ,  $n = 1$ ,  $k = j$  and  $m = 1/2$ , the explicit and implicit stages of the R11 method are obtained, respectively.

### 3 Convergence

When numerically solving a PDE, it is necessary that the numerical solution approaches the analytical solution (FORTUNA, 2012). To obtain the approximate solution, the number of operations depends on the  $\Delta x$  and  $\Delta t$  partitions. From a computational point of view, the lower the values of  $\Delta x$  and  $\Delta t$ , the closer the numerical solution will be to the analytical solution; however, the number of operations done by the computer rises. This approximation process can generate an accumulation of uncontrolled errors, characterizing a numerical method as stable or unstable.

To ensure that the numerical solution is as close as possible to the real solution of the problem, it is necessary and sufficient that the conditions established by Lax's theorem are satisfied (FORTUNA, 2012, CUMINATO; MENEGUETTE, 2013), that is:

**Theorem 3.1.** *A necessary and sufficient condition for the convergence of a numerical method, when applied to a well-posed initial value problem, is that the discretization scheme is consistent and stable.*

In this context, this section presents the theory of consistency and stability of numerical methods.

#### 3.1 Consistency

When the derivatives of a PDE are replaced by finite difference formulae, a finite difference equation (FDE) and an associated error are obtained, so that the PDE analytical solution does not satisfy the FDE precisely. This error is defined as Local Truncation Error (LTE) and appears due to the use of a finite amount of terms while expanding a function in Taylor series (FORTUNA, 2012, CUMINATO; MENEGUETTE, 2013).

To show that a FDE is consistent with PDE, consider the  $\Delta x, \Delta t \rightarrow 0$  partitions. If LTE also tends to zero, then FDE is said to be consistent with PDE (FORTUNA, 2012, CUMINATO; MENEGUETTE, 2013).

### 3.2 Stability

A numerical method is said to be stable when errors and disturbances in the solution are not amplified endlessly. One of the most used techniques in the study of the stability of finite difference equations is the Von Neumann criterion, which consists of expanding the Local Truncation Error  $E_i^n$  in Fourier series, as follows:

$$E_i^n = \sum_j \phi_j^n e^{IQx_i}, \quad (6)$$

where  $I = \sqrt{-1}$  and for the  $j$ -th component,  $\phi_j^n$  is its amplitude at time  $n$  and  $Q$  its wave number (CUMINATO; MENEGUETTE, 2013).

## 4 Convergence Analysis

In this section, the consistency and stability of the R11 method are checked. For that, Taylor series expansion is employed to show consistency and Von Neumann criterion to determine the stability region, as discussed in subsection 3. 1 and 3. 2, respectively. For the other methods, Explicit Euler and Crank-Nicolson, the convergence analysis is reviewed in Fortuna (2012), and Cuminato and Meneguette (2013).

### 4.1 Consistency of the explicit stage of the R11 method

The consistency analysis presented in this subsection is performed for the explicit stage of the R11 method. For the implicit stage, the analysis is analogous and will not be presented.

Without loss of generality, consider in the equation (1) the thermal diffusivity term  $\alpha(x, t)$  as constant and the source term  $r(x, t)$  equal to zero, that is:

$$u_t = \alpha u_{xx}. \quad (7)$$

The explicit stage of the R11 method is given by:

$$U_i^{j+\frac{1}{2}} = (1 - \gamma)U_i^j + \frac{\gamma}{2} \left( U_{i-1}^j + U_{i+1}^j \right), \quad (8)$$

with  $\gamma = \frac{\alpha \Delta t}{(\Delta x)^2}$ .

Expanding in Taylor series the spatial component of  $U_{i+1}^j$  and  $U_{i-1}^j$ , given in (8), we obtain:

$$U_{i\pm 1}^j = U_i^j \pm \Delta x \frac{\partial}{\partial x} U_i^j \pm \frac{\Delta x^2}{2!} \frac{\partial^2}{\partial x^2} U_i^j \pm \frac{\Delta x^3}{3!} \frac{\partial^3}{\partial x^3} U_i^j \pm \mathcal{O}(\Delta x^4), \quad (9)$$

so that  $\mathcal{O}(\Delta x^4)$ , represents the upper and equal terms to the fourth order in the expansion in Taylor series.

Similarly, performing the Taylor series expansion in the time component of  $U_i^{j+\frac{1}{2}}$ , given in (8), it becomes:

$$U_i^{j+\frac{1}{2}} = U_i^j + \frac{\Delta t}{4} \frac{\partial}{\partial t} U_i^j + \frac{\Delta t}{4} \frac{\partial}{\partial t} \left( U_i^j + \frac{\Delta t}{2} \frac{\partial}{\partial t} U_i^j \right) + \mathcal{O} \left( \frac{\Delta t}{2} \right)^3, \quad (10)$$

so that  $\mathcal{O} \left( \frac{\Delta t}{2} \right)^3$  represents the upper and equal terms of the third order in the expansion in Taylor series.

Yet, the expansion (10) deserves further consideration. Note that the term  $U_i^{j+\frac{1}{2}}$  can be rewritten in the form

$$U_i^{j+\frac{1}{2}} \approx U_i^j + \frac{\Delta t}{2} \frac{\partial}{\partial t} U_i^j, \quad (11)$$

which comes from the discretization of the temporal term by the regressive finite difference formula, at time levels  $j$  e  $j + \frac{1}{2}$ . Thus, replacing (11) in (10) we get the approximate equation:

$$U_i^{j+\frac{1}{2}} = U_i^j + \frac{\Delta t}{4} \frac{\partial}{\partial t} U_i^j + \frac{\Delta t}{4} \frac{\partial}{\partial t} U_i^{j+\frac{1}{2}} + \mathcal{O} \left( \frac{\Delta t}{2} \right)^3. \quad (12)$$

On the other hand, applying the finite difference formula centred on the term  $\frac{\partial}{\partial t} U_i^{j+\frac{1}{2}}$  of (12), encounter:

$$\frac{\partial}{\partial t} U_i^{j+\frac{1}{2}} \approx \frac{U_i^{j+1} - U_i^j}{\Delta t}. \quad (13)$$

In addition, the term  $U_i^{j+\frac{1}{2}}$  is an arithmetic mean between the time levels  $j$  and  $j + 1$ , thus  $U_i^{j+\frac{1}{2}} = \frac{U_i^{j+1} + U_i^j}{2}$ , so  $U_i^{j+1} = 2U_i^{j+\frac{1}{2}} - U_i^j$ . Substituting the equality of  $U_i^{j+1}$  in (13), we have:

$$\frac{\partial}{\partial t} U_i^{j+\frac{1}{2}} \approx \frac{2U_i^{j+\frac{1}{2}} - 2U_i^j}{\Delta t}. \quad (14)$$

Replacing (14) in (12), the approximate equation follows:

$$U_i^{j+\frac{1}{2}} = U_i^j + \frac{\Delta t}{4} \frac{\partial}{\partial t} U_i^j + \frac{\Delta t}{4} \left( \frac{2U_i^{j+\frac{1}{2}} - 2U_i^j}{\Delta t} \right) + \mathcal{O}\left(\frac{\Delta t}{2}\right)^3. \quad (15)$$

Grouping the common terms in (15), and simplifying equation:

$$U_i^{j+\frac{1}{2}} = U_i^j + \frac{\Delta t}{2} \frac{\partial}{\partial t} U_i^j + 2\mathcal{O}\left(\frac{\Delta t}{2}\right)^3. \quad (16)$$

Finally, replacing the expansions (9) and (16) in (8), as well as some algebraic manipulations, we obtain:

$$\underbrace{\frac{\partial}{\partial t} U_i^j - \alpha \frac{\partial^2}{\partial x^2} U_i^j}_{\text{PDE}} = \underbrace{-4\mathcal{O}\left(\frac{\Delta t}{2}\right)^2 + 2\alpha\mathcal{O}(\Delta x)^2}_{\text{LTE}}. \quad (17)$$

Making  $\Delta x, \Delta t \rightarrow 0$  in (17), the LTE tends to zero, leaving only the finite difference equation applied to a point in the mesh, that is

$$\frac{\partial}{\partial t} U_i^j = \alpha \frac{\partial^2}{\partial x^2} U_i^j. \quad (18)$$

Therefore, from (18), it is concluded that the first stage of the R11 method is consistent.

## 4.2 Stability of the R11 method

To find the stability region of the R11 method, the Von Neumann criterion is used, as discussed in the subsection 3. 2.

Consider the explicit stage of the R11 method, given by

$$U_i^{j+\frac{1}{2}} = (1 - \gamma)U_i^j + \frac{\gamma}{2} \left( U_{i-1}^j + U_{i+1}^j \right), \quad (19)$$

with  $\gamma = \frac{\alpha\Delta t}{(\Delta x)^2}$ .

Initially, each term of (19) is replaced by its equivalent Fourier series, like (6), that is

$$U_i^{j+\frac{1}{2}} = \phi^{j+\frac{1}{2}} e^{IQx_i}, \quad (20)$$

$$U_i^j = \phi^j e^{IQx_i}, \quad (21)$$

$$U_{i\pm 1}^j = \phi^j e^{IQ(x_i \pm \Delta x)} = \phi^j e^{IQx_i} e^{\pm IQ\Delta x}. \quad (22)$$

Substituting (20)-(22) in (19), we have

$$\phi^{j+\frac{1}{2}}e^{IQx_i} = (1 - \gamma)\phi^j e^{IQx_i} + \frac{\gamma}{2} (\phi^j e^{IQx_i} e^{-IQ\Delta x} + \phi^j e^{IQx_i} e^{IQ\Delta x}). \quad (23)$$

As  $e^{IQ\Delta x} + e^{-IQ\Delta x} = 2 \cos(Q\Delta x)$ , performing some algebraic manipulations in (23), it follows that:

$$\frac{\phi^{j+\frac{1}{2}}}{\phi^j} = 1 + \gamma(\cos(Q\Delta x) - 1). \quad (24)$$

Establishing the criterion of stability in (24):

$$\left| \frac{\phi^{j+\frac{1}{2}}}{\phi^j} \right| = \left| 1 + \gamma(\cos(Q\Delta x) - 1) \right| \leq 1, \quad (25)$$

that is,

$$1 + \gamma(\cos(Q\Delta x) - 1) \leq 1, \quad (26)$$

$$1 + \gamma(\cos(Q\Delta x) - 1) \geq -1. \quad (27)$$

The inequality (26) is always satisfied, because when considering the maximum and minimum points of  $\cos(Q\Delta x)$ , we find  $\gamma > 0$ . On the other hand, from the second inequality (27), assuming the maximum and minimum points of  $\cos(Q\Delta x)$ , we find  $\gamma \leq 1$ . Thus, a condition is initially found in the explicit stage is  $0 < \gamma \leq 1$ .

Subsequently, we must analyze the stability conditions for the implicit stage of the R11 method, given by:

$$U_i^{j+1} = U_i^{j+\frac{1}{2}} + \frac{\gamma}{2} (U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}), \quad (28)$$

with  $\gamma = \frac{\alpha\Delta t}{(\Delta x)^2}$ .

Replacing each term of the implicit stage of the R11 method by its equivalent Fourier series, as (6), we obtain:

$$U_i^{j+\frac{1}{2}} = \phi^{j+\frac{1}{2}} e^{IQx_i}, \quad (29)$$

$$U_i^{j+1} = \phi^{j+1} e^{IQx_i}, \quad (30)$$

$$U_{i\pm 1}^{j+1} = \phi^{j+1} e^{IQ(x_i \pm \Delta x)} = \phi^{j+1} e^{IQx_i} e^{\pm IQ\Delta x}. \quad (31)$$

Replacing (29)-(31) in (28), it follows that:

$$\phi^{j+1} e^{IQx_i} = \phi^{j+\frac{1}{2}} e^{IQx_i} + \frac{\gamma}{2} \left( \phi^{j+1} e^{IQx_i} e^{-IQ\Delta x} - 2\phi^{j+1} e^{IQx_i} + \phi^{j+1} e^{IQx_i} e^{IQ\Delta x} \right). \quad (32)$$

As  $e^{IQ\Delta x} + e^{-IQ\Delta x} = 2 \cos(Q\Delta x)$ , while some algebraic manipulations in (32), we obtain:

$$\frac{\phi^{j+\frac{1}{2}}}{\phi^{j+1}} = 1 - \gamma \left( \cos(Q\Delta x) - 1 \right). \quad (33)$$

Establishing the stability criterion at (33):

$$\left| \frac{\phi^{j+\frac{1}{2}}}{\phi^{j+1}} \right| = \left| 1 - \gamma \left( \cos(Q\Delta x) - 1 \right) \right| \leq 1, \quad (34)$$

that is,

$$1 - \gamma (\cos(Q\Delta x) - 1) \leq 1, \quad (35)$$

$$1 - \gamma (\cos(Q\Delta x) - 1) \geq -1. \quad (36)$$

The inequalities (35) and (36) are always true when analyzing the maximum and minimum points of  $\cos(Q\Delta x)$ . Thus, for the R11 method to be a stable method, one must consider the conditions imposed in the explicit stage. Therefore, according to Lax theorem, the R11 method is convergent, as long as the condition  $0 < \frac{\alpha\Delta t}{(\Delta x)^2} \leq 1$  is respected.

Finally, the stability conditions to Explicit Euler, Crank-Nicolson and R11 methods are evaluated by following expressions:

$$\Phi_{\text{Exp-Euler}} = 1 + 2\gamma(\cos(Q\Delta x) - 1) \quad (\text{Explicit Euler}) \quad (37)$$

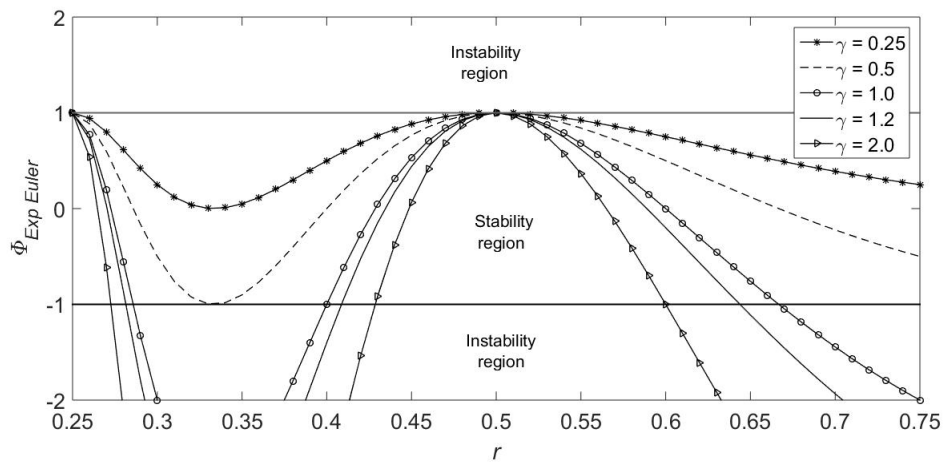
$$\Phi_{\text{CN}} = \frac{1 + \gamma(\cos(Q\Delta x) - 1)}{1 - \gamma(\cos(Q\Delta x) - 1)} \quad (\text{Crank-Nicolson}) \quad (38)$$

$$\Phi_{\text{R11}} = 1 + \gamma(\cos(Q\Delta x) - 1) \quad (\text{R11}) \quad (39)$$

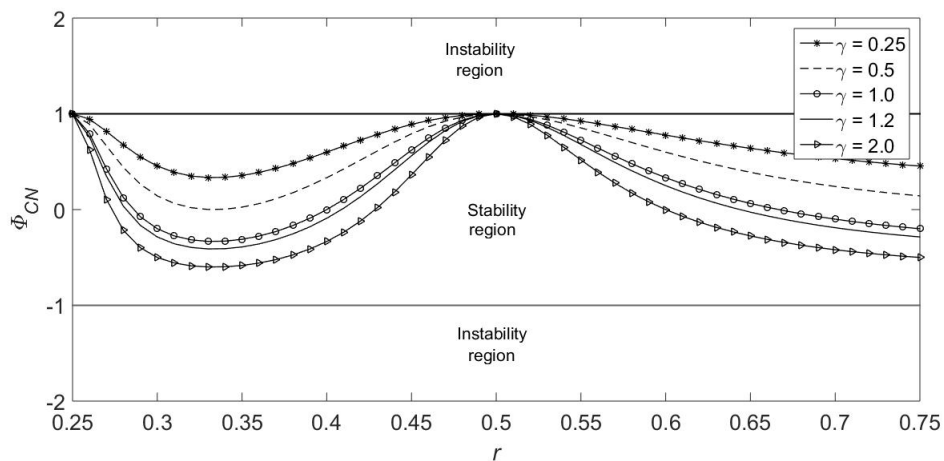
Table 1 shows, the stability criteria of the Explicit Euler, Crank-Nicolson and R11 methods, as well as their convergence orders.

As indicated in Table 1 the Explicit Euler and R11 methods are conditionally stable, while the Crank-Nicolson method is unconditionally stable. Figure 1 illustrates the stability region of these methods in the interval  $r = [0.25, 0.75]$ .

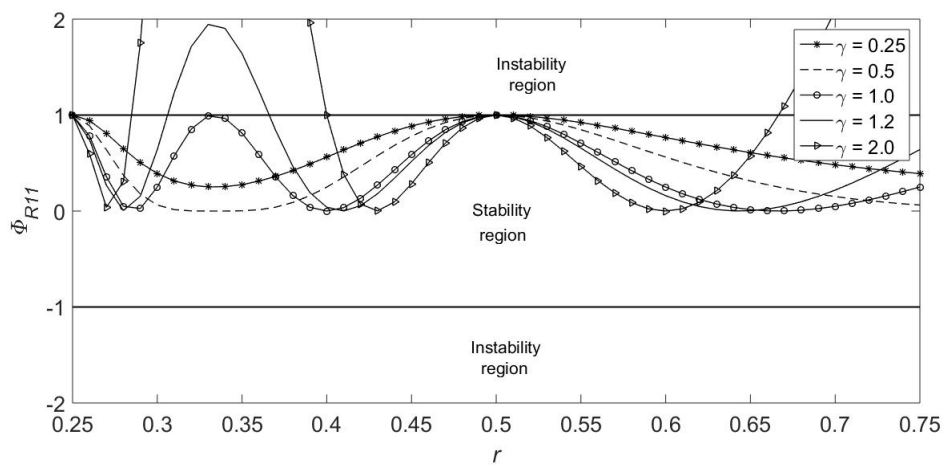
Figure 1 – Stability region: (a) Explicit Euler method; (b) Crank-Nicolson method; (c) R11 method.



(a)



(b)



(c)

Source: The authors.

Table 1 – Stability interval and order of convergence of the methods.

<b>R11 method</b>		<b>Crank-Nicolson method</b>		<b>Explicit Euler method</b>	
Stability	Convergence order	Stability	Convergence order	Stability	Convergence order
$\gamma \in (0, 1]$	$\mathcal{O}(\Delta t/2)^2$ $\mathcal{O}(\Delta x)^2$	$\gamma \in (0, \infty)$	$\mathcal{O}(\Delta t)^2$ $\mathcal{O}(\Delta x)^2$	$\gamma \in (0, 0.5]$	$\mathcal{O}(\Delta t)$ $\mathcal{O}(\Delta x)^2$

Source: The authors.

## 5 Results

All the results presented in this section come from the application of the numerical methods previously presented in model (1)-(3), for different initial and boundary conditions, in which the numerical solutions will be compared with the analytical solutions of the following examples. The spatial and temporal convergence orders of the methods used will also be presented.

### 5.1 Example 1

Considering the domain  $0 \leq x \leq 1$  and  $0 \leq t \leq 1$ , the one-dimensional model of heat diffusion (1)-(3), where the initial condition

$$u(x, 0) = 100\text{sen}(\pi x), \tag{40}$$

and the boundary conditions of the Dirichlet type

$$u(0, t) = 0, \tag{41}$$

$$u(1, t) = 0,$$

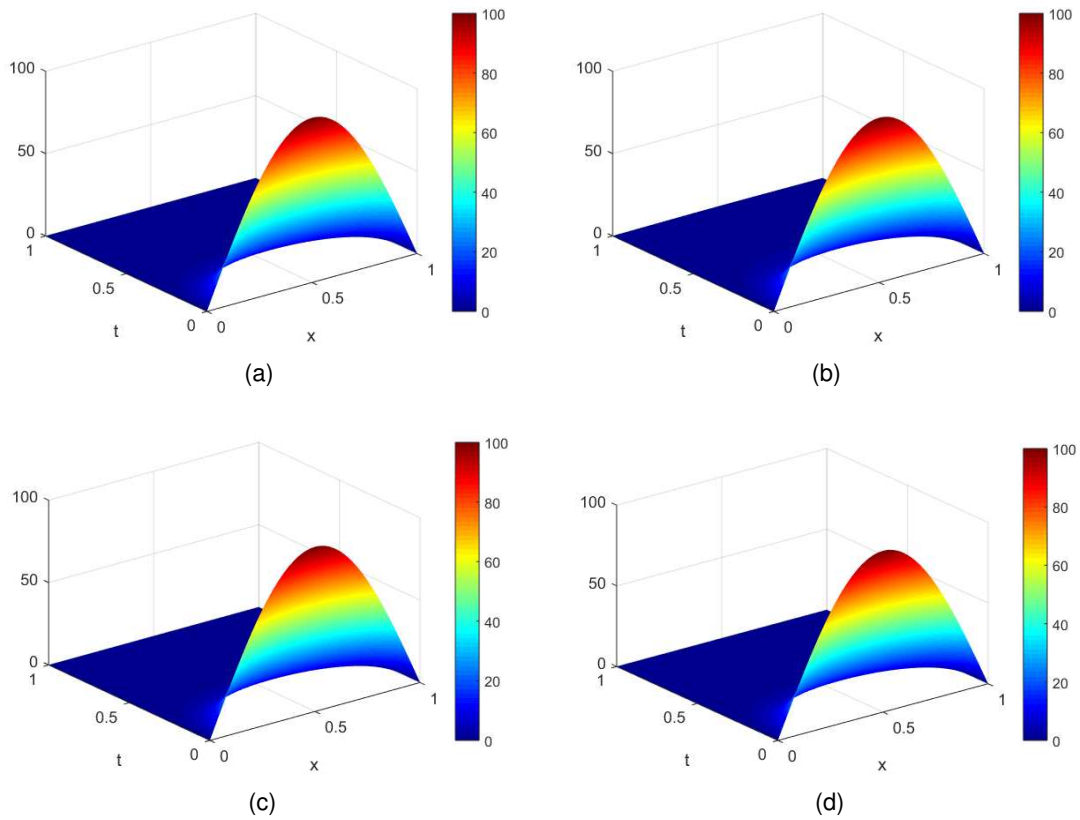
have the analytical solution given by (CUMINATO; MENEGUETTE, 2013):

$$u(x, t) = 100e^{-\pi^2 t} \text{sen}(\pi x). \tag{42}$$

Thus, given the equation (1), and using the initial and boundary conditions, equations (40)-(41), Figure 2 presents the solutions obtained by the numerical methods: Euler Explicit, Crank-Nicolson and R11, along with the analytical solution, when  $M_x = 40$ ,  $M_t = 3200$  and  $\gamma = 0.5$ .

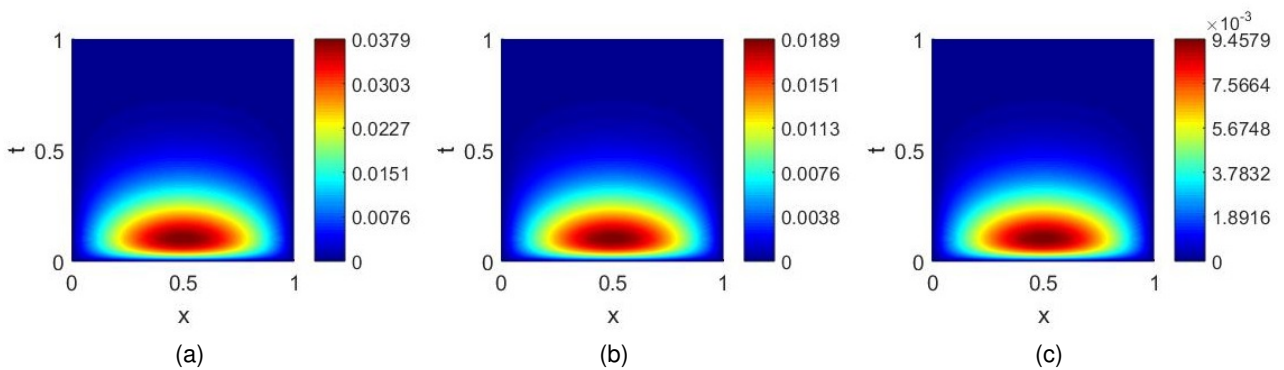
From Figure 2(a)-(d), it is observed that the numerical solutions present, qualitatively, the same behavior of the analytical solution. As a way of assessing how close the numerical solutions are to the analytical solution, Figure 3 shows the behavior of the approximation errors of the numerical methods.

Figure 2 – Analytical and numerical solutions: (a) Analytical; (b) Explicit Euler; (c) Crank-Nicolson; (d) R11 method.



Source: The authors.

Figure 3 – Approximation errors of numerical methods in the problem domain: (a) Explicit Euler method; (b) Crank-Nicolson method; (c) R11 method.



Source: The authors.

From the results presented in Figure 3, It is observed that the numerical methods Euler Explicit, Figure 3(a), Crank-Nicolson, Figure 3(b) and R11, Figure 3(c), reached the following maximum approximation error values:  $3.7864 \times 10^{-2}$ ,  $1.8882 \times 10^{-2}$  and  $9.4579 \times 10^{-3}$ , respectively.

The R11 method had the lowest error values, while the Explicit Euler method had the highest values. Note that the results presented were obtained for  $\gamma = 0.5$ , that is, at the limit of the stability region of the Explicit Euler method (FORTUNA, 2012, CUMINATO; MENEGUETTE, 2013).

For a more detailed assessment of the characteristics of the methods employed, we set  $M_x = 10$  and vary  $M_t$ , so that  $\gamma$  is within the stability region. The results are presented in Table 2 for the point  $x = 0.5$ , at the time level  $t = 1$ . The relative error  $E_{ij}$  for the calculation of the temporal convergence order, denoted by  $p_t$ , and the processing time of the solutions are also displayed.

Table 2 – Numerical solutions for  $M_x = 10$ , in  $x = 0.5$  and  $t = 1$  for several  $M_t$  values. The analytical solution is  $u(0.5, 1) = 5.1723 \times 10^{-3}$ .

$\gamma$	$M_t$	$U_{ij}$	$E_{ij}$	$p_t$	<b>cpu (s)</b>
<b>Explicit Euler method</b>					
0.5	200	$4.3779 \times 10^{-3}$	$4.6106 \times 10^{-1}$	0.90194	0.11427
0.25	400	$4.9653 \times 10^{-3}$	$4.0020 \times 10^{-2}$	0.95192	0.10055
0.125	800	$5.2796 \times 10^{-3}$	$2.0745 \times 10^{-2}$	0.97619	0.13868
0.0625	1600	$5.4421 \times 10^{-3}$	$5.2162 \times 10^{-2}$	0.98816	0.15446
0.03125	3200	$5.5247 \times 10^{-3}$	$6.8132 \times 10^{-2}$		0.21561
0.015625	6400	$5.5663 \times 10^{-3}$	$7.6175 \times 10^{-2}$		0.34460
<b>R11 method</b>					
1	100	$4.3779 \times 10^{-3}$	$1.5359 \times 10^{-1}$	1.8332	0.63517
0.5	200	$4.9653 \times 10^{-3}$	$4.0020 \times 10^{-2}$	1.9184	0.30289
0.25	400	$5.2796 \times 10^{-3}$	$2.0745 \times 10^{-2}$	1.9596	0.47587
0.125	800	$5.4421 \times 10^{-3}$	$5.2162 \times 10^{-2}$	1.9799	1.45370
0.0625	1600	$5.5247 \times 10^{-3}$	$6.8132 \times 10^{-2}$	1.9899	20.3880
0.03125	3200	$5.5663 \times 10^{-3}$	$7.6180 \times 10^{-2}$		37.0212
0.015625	6400	$5.5872 \times 10^{-3}$	$8.0215 \times 10^{-2}$		71.0903
<b>Crank-Nicolson method</b>					
2	50	$5.4345 \times 10^{-3}$	$5.0693 \times 10^{-2}$	1.9866	0.11617
1	100	$5.5645 \times 10^{-3}$	$7.5827 \times 10^{-2}$	1.9967	0.10966
0.5	200	$5.5972 \times 10^{-3}$	$8.2149 \times 10^{-2}$	1.9992	0.33389
0.25	400	$5.6055 \times 10^{-3}$	$8.3754 \times 10^{-2}$	1.9998	0.21327
0.125	800	$5.6075 \times 10^{-3}$	$8.4141 \times 10^{-2}$	1.9999	0.20511
0.0625	1600	$5.6081 \times 10^{-3}$	$8.4257 \times 10^{-2}$	2.0000	0.30775
0.03125	3200	$5.6081 \times 10^{-3}$	$8.4257 \times 10^{-2}$		0.76860
0.015625	6400	$5.6082 \times 10^{-3}$	$9.8544 \times 10^{-2}$		1.76060

Source: The authors.

Similarly,  $M_t = 25600$  is fixed and  $M_x$  is variable. The results are presented in Table 3, as well as the relative error for the calculation of the spatial convergence order, denoted by  $p_x$ , and the processing time of the solutions.

Table 3 – Numerical solutions for  $M_t = 25600$ , in  $x = 0.5$  and  $t = 1$  for several  $M_x$  values. The analytical solution is  $u(0.5, 1) = 5.1723 \times 10^{-3}$ .

$\gamma$	$M_x$	$U_{ij}$	$E_{ij}$	$p_x$	cpu (s)
<b>Explicit Euler method</b>					
0.0039	10	$5.5977 \times 10^{-3}$	$1.5558 \times 10^{-1}$	2.0505	0.11427
0.0156	20	$5.2682 \times 10^{-3}$	$1.8541 \times 10^{-2}$	2.0126	0.10055
0.0625	40	$5.1887 \times 10^{-3}$	$3.1707 \times 10^{-3}$		0.13868
0.2500	80	$5.1690 \times 10^{-3}$	$6.3801 \times 10^{-4}$		0.15446
<b>R11 method</b>					
0.0039	10	$5.6029 \times 10^{-3}$	$8.3256 \times 10^{-2}$	2.0499	594
0.0156	20	$5.2733 \times 10^{-3}$	$1.0639 \times 10^{-2}$	2.0073	1287
0.0625	40	$5.1937 \times 10^{-3}$	$4.1374 \times 10^{-3}$	2.0000	2530
0.2500	80	$5.1739 \times 10^{-3}$	$3.1719 \times 10^{-4}$		5018
1.0000	160	$5.1690 \times 10^{-3}$	$6.3801 \times 10^{-4}$		14849
<b>Crank-Nicolson method</b>					
0.0039	10	$5.6082 \times 10^{-3}$	$8.4276 \times 10^{-2}$	2.0516	190
0.0156	20	$5.2782 \times 10^{-3}$	$2.0474 \times 10^{-2}$	2.0073	459
0.0625	40	$5.1986 \times 10^{-3}$	$5.0848 \times 10^{-3}$	2.0000	1419
0.2500	80	$5.1788 \times 10^{-3}$	$1.2567 \times 10^{-3}$		4464
1.0000	160	$5.1735 \times 10^{-3}$	$2.3201 \times 10^{-4}$		18753

Source: The authors.

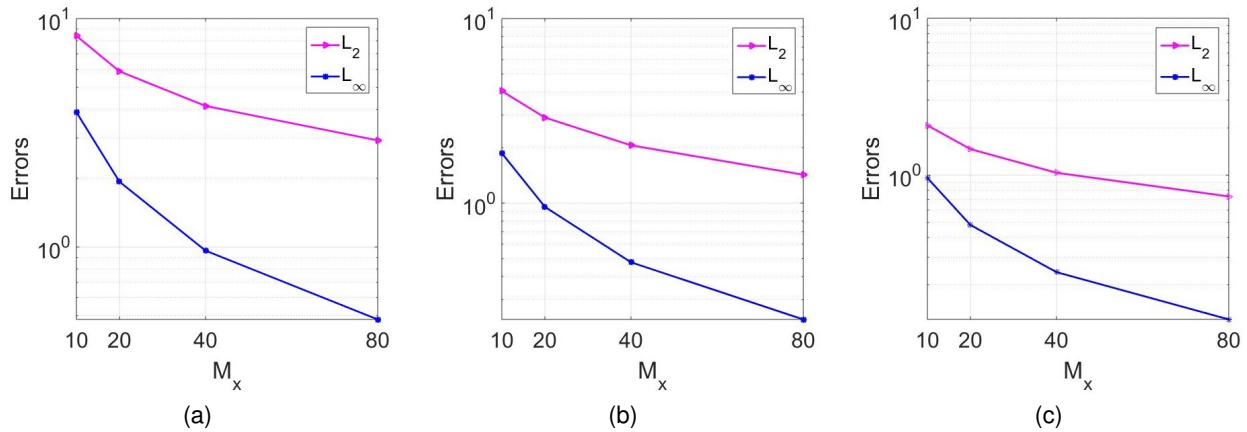
The R11 method has a greater stability region than the Explicit Euler method. In this context, while the Explicit Euler method obtained the result of  $U(0.5, 1) = 4.3779 \times 10^{-3}$  (see Table 2) for a temporal refinement of  $M_t = 200$ , the R11 method obtained the same result; however, for a minor temporal refinement, that is,  $M_t = 100$ . The Crank-Nicolson method, on the other hand, presented results close to the analytical solution, obtaining relative errors in the order of  $10^{-2}$ , although the approximation error increases as time is refined. This propagation of errors, in general, is corrected when space is also refined, according to the data presented in Table 3.

Regarding the estimated order of convergence  $p_t$  and  $p_x$ , it appears that they are consistent with the order of analytical convergence as shown in Table 1, that is, both Crank-Nicolson and R11 method have a second-order in time and space, while the Explicit Euler method has a first-order in

time and the second-order in space. Also, in all analyses, the Explicit Euler method obtained the shortest computational time, since this method does not require the resolution of a system of linear equations.

The analyses presented above were performed for a single specific point in the mesh. Now, to analyze the influence of all points of the mesh in obtaining the results, the spatial domain is discretized to  $M_x = 10, 20, 40$  and  $80$ , which represent  $\Delta x = 0.1, 0.05, 0.025, 0.0125$ , respectively, and the relative errors  $L_2$  and  $L_\infty$  are calculated on the different meshes. The numerical results Figure 4 improve considerably when the number of elements in the mesh is increased.

Figure 4 – Errors in norms  $L_2$  and  $L_\infty$ : (a) Explicit Euler method; (b) Crank-Nicolson method; (c) R11 method.



Source: The authors.

The R11 method produces numerical solutions with fewer error, as previously analyzed at point  $x = 0.5$  in  $t = 1$ , with  $\gamma$  within its stability region. From the results presented in Figure 3, it is possible to indicate that this situation extends to all points of the mesh.

## 5.2 Example 2

As shown in Example 1, the R11 method produces numerical solutions that are closer to the analytical solution. In this example, it will be analyzed for which values of  $\gamma$  this method produces fewer numerical errors. For this, the model (1) is considered with the initial condition

$$u(x, 0) = \left(x - \frac{1}{3}\right)^6, \tag{43}$$

and the boundary conditions

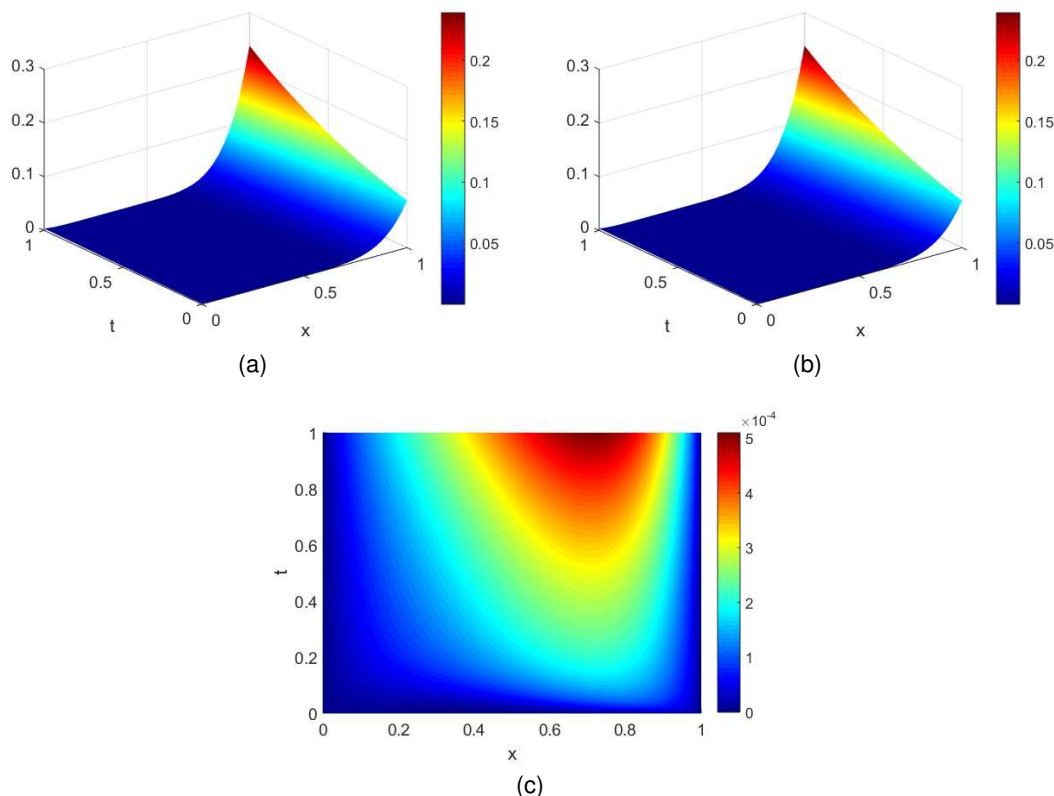
$$\begin{aligned} u(0, t) &= 0.0014e^t, \\ u(1, t) &= 0.0878e^t. \end{aligned} \tag{44}$$

The analytical solution to model (1), satisfying the conditions (43)-(44), is given by (GAO; SUN, 2012):

$$u(x, t) = \left(x - \frac{1}{3}\right)^6 e^t. \tag{45}$$

Similarly, considering the domain  $0 \leq x \leq 1$  e  $0 \leq t \leq 1$ , Figure 5(a)-(b) presents the analytical solution (45) and numerical solutions obtained by the R11 method, along with the approximation errors, Figure 5(c), for  $M_x = 40$ ,  $M_t = 3200$ , and  $\gamma = 0.5$ .

Figure 5 – Solutions to one-dimensional heat diffusion equation (1) with initial and boundary conditions given by (43)-(44): (a) Analytical solution, equation (45); (b) R11 method; (c) Approximation errors of R11 method in the problem domain.

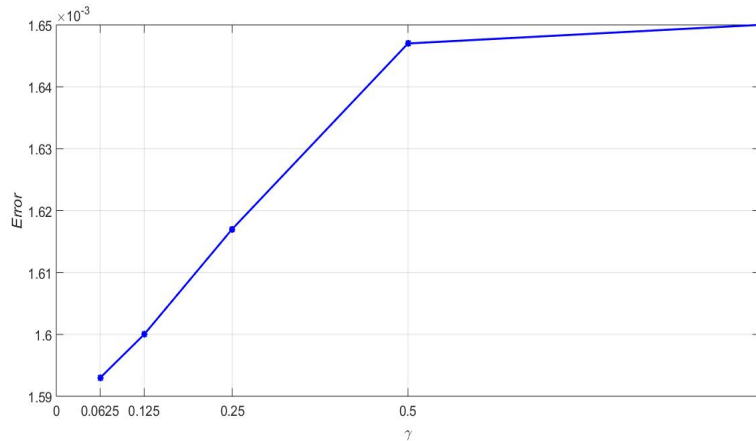


Source: The authors.

In regard to Figure 5, the R11 method produced similar results to the analytical solution's, with an absolute error in the order of  $10^{-4}$ . To check for which values of  $\gamma$  the R11 method produces

fewer errors, we set  $M_x = 20$  and vary  $M_t$ . The respective values of  $\gamma$  and the error at point  $x = 0.5$  in  $t = 1$  are shown in Figure 6.

Figure 6 – Behavior of the error produced by the R11 method, in the solution of the model (1), (43) and (44), for different values of  $\gamma$ .



Source: The authors.

In Figure 6 the error is decreasing proportionally with  $\gamma$  values, that is, the lower the value of  $\gamma$ , fewer the error. On the other hand, when  $\gamma$  approaches the value that delimits its region of stability, the error of R11 method increase.

## 6 Conclusions

This article aimed to verify which of the methods: Euler Explicit, Crank-Nicolson and R11, produced results with lower numerical errors when used in the one-dimensional heat diffusion equation for different initial and boundary conditions.

In the examples presented, it was found that the R11 method was the numerical method that produced solutions with the least approximation error. When comparing this method with the others, it halves the number of calculations performed to obtain the numerical solution. This is due to the characteristic of the method that calculates the solution using two stages, that is, it uses time intervals  $\Delta t/2$ , therefore, the time refinement is less than the refinement of the other methods. On the other hand, the R11 method, as well as the Crank-Nicolson method, requires the iterative resolution of a system of linear equations. Thus, the processing time of its numerical solutions is longer when compared to the processing time of the Explicit Euler method.

The numerical approach conducted in this article did not require much computational effort in processing the solutions. In this context, it is concluded that, for the evaluated examples, the R11 method proved to be superior to the Explicit Euler and Crank-Nicolson methods.

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