Numerical study of different methods applied to the one-dimensional transient heat equation

Estudo numérico de diferentes métodos aplicados à equação transiente do calor unidimensional

Neyva Maria Lopes Romeiro
State University of Londrina (UEL), Department of Math and PGMAC, Londrina, PR, Brazil
https://orcid.org/0000-0002-3249-3490, `nromeiro@uel.br`

Eduardo Oliveira Belinelli
Federal University of Paraná (UFPR), PPGMNE, Curitiba, PR, Brazil
https://orcid.org/0000-0002-5925-086X, `eduardo.belinelli@ufpr.br`

Jesika Maganin
Federal University of Paraná (UFPR), PPGMNE, Curitiba, PR, Brazil
https://orcid.org/0000-0003-2719-3124, `jesikamaganin@ufpr.br`

Paulo Laerte Natti
State University of Londrina (UEL), Department of Math and PGMAC, Londrina, PR, Brazil
https://orcid.org/0000-0002-5988-2621, `plnatti@uel.br`

Eliandro Rodrigues Cirilo
State University of Londrina (UEL), Department of Math and PGMAC, Londrina, PR, Brazil
https://orcid.org/0000-0001-7530-1770, `ercirilo@uel.br`

Article informations

How to cite this article

Abstract
This article aims to compare the results obtained by applying three numerical methods: Explicit Euler, Crank-Nicolson, and Multi-stage (R11), in the one-dimensional heat diffusion transient equation with different initial and boundary conditions. The discretization process was performed using the finite difference method. In order to guarantee the convergence of the methods used, consistency and stability were verified by Lax theorem. The results are presented in graphs and tables that contain the data of the analytical solution and the numerical solutions. It was observed that the results obtained by R11 method generated solutions with minor errors.

Palavras-Chave
Modelo Matemático
Método de Diferenças Finitas
Convergência

Resumo
Este artigo tem por objetivo comparar os resultados obtidos pela aplicação de três métodos numéricos: Euler Explícito, Crank-Nicolson e Multi-estágio (R11), na equação transiente da difusão do calor unidimensional com dife-rentes condições iniciais e de contorno. O processo de discretização foi realizado pelo método de diferenças finitas. Para garantir a convergência dos métodos utilizados foi verificada a consistência e a estabilidade pelo Teorema de Lax. Os resultados são apresentados em gráficos e tabelas que contêm dados da solução analítica e das soluções numéricas. Observou-se que os resultados obtidos pelo método R11 gerou soluções com menores erros.

1   Introduction

The transient heat diffusion equation is a linear partial differential equation (PDE) of first-order in time and second-order in space. Transient problems involve the temporal variation of the physical quantities of interest. From the initial and contour values of these quantities,the PDE solution calculates their values in successive time intervals (FORTUNA, 2012).

In this article, we consider the problem of finding numerical solutions for the one-dimensional heat diffusion equation in transient regime with different initial and boundary conditions, described mathematically by

$$u_t - \alpha(x,t)u_{xx} = r(x,t) \qquad u(x,t) \in \Omega \times (0,T) \qquad (1)$$

$$u(x,0) = f(x) \qquad (2)$$

$$u(x,t) = \phi(x,t) \qquad (x,t) \in \partial\Omega \times (0,T), \qquad (3)$$

so that $\alpha(x,t)$ represents thermal diffusivity (BOYCE; DIPRIMA, 1985, FORTUNA, 2012), $\Omega = [0,L]$ is the spatial domain, and $(0,T)$ is the temporal domain. The functions $r(x,t)$, $f(x)$ and $\phi(x,t)$ describe the source term and the initial and boundary conditions, respectively.

In recent years, several computational techniques were developed to obtain numerical solutions of PDEs. This approach consists of transforming continuous problems into discrete problems, involving both the equation of interest and the geometric domain of solutions. In the latter, the transformation consists of the decomposition of a continuous region into a finite set of points. Then, the

equations of interest must be written in the form of arithmetic operations, depending on the discretized domain.

Several methods in the literature can be used to obtain numerical solutions to PDEs, such as the methods developed in Horváth (2002), Kadalbajoo and Awasthi (2006), Wang, Nakagawa and Yamamoto (2010), Ladeia *et al.* (2013), Gu *et al.* (2018), Hajipour *et al.* (2018), Kazen and Dehghan (2018) and Saita *et al.* (2018). In this article, three methods will be particularly addressed: the Explicit Euler method, Crank-Nicolson (CN), and Multi-stage (R11), which are found in Faragó and Palencia (2002), Araújo and Márquez (2012) and Pereira, Lisboa and Dias Filho (2017). The objective is to verify which method produces numerical solutions closer to analytical solutions. Based on the results obtained, error analysis will be carried out, as well as the time taken to process the solutions. The convergence of the methods is verified through Lax theorem.

The article is accomplished as follows: in section 2, the model discretization process, equation (1), is carried out using the finite difference method. Section 3 presents the convergence theory of numerical methods. In section 4, the convergence analysis (consistency and stability) of the R11 method is verified. Section 5 presents the results obtained in this article. Finally, some considerations are drawn in section 6.

2   Numerical model

The geometric domain of solutions, defined in this article, is a region of length $[0, L]$ in the spatial domain x, and an interval $(0, T)$ in the temporal domain t. Dividing the interval $(0, L)$ into M_x equal parts in length Δx, there is $M_x + 1$ points $x_i = i \Delta x$, with $i = 1, \ldots, M_x$, so that $\Delta x = L/M_x$. Similarly, dividing the interval $(0, T)$ into M_t equal parts in length Δt, it results in $M_t + 1$ points $t_j = j \Delta t$, with $j = 1, \ldots, M_t$, so that $\Delta t = T/M_t$. Thus, the approximations of the differential terms in equation (1) must be performed according to the points of the discretized geometric domain defined by the set:

$$(x_i, t_j) = (x_0 + i \Delta x, y_0 + j \Delta t) \quad \text{with} \ \ i = 1, \ldots, M_x \ \text{and} \ \ j = 1, \ldots, M_t, \tag{4}$$

in which (x_0, y_0) represents the origin of the geometric domain of solutions.

In this article, u_{ij} is denoted as the analytical solution at point (x_i, t_j) and for U_{ij} as the numerical approximate value of u_{ij}.

The equation (1) is discretized by the finite difference method, given by

$$\frac{U_i^{j+n} - U_i^{k+m}}{\Delta t/s} = \sigma\left(\alpha\frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{\Delta x^2}\right) + (1-\sigma)\left(\alpha\frac{U_{i-1}^{j} - 2U_i^{j} + U_{i+1}^{j}}{\Delta x^2}\right) + r_i^j. \qquad (5)$$

In the model discretization process, $\alpha(x,t)$ was considered constant. In (5), in case that $\sigma = 0$, $s = 1$, $n = 1$, $k = j$ and $m = 0$, the Explicit Euler method is obtained. If $\sigma = 1/2$, $s = 1$, $n = 1$, $k = j$ and $m = 0$, the Crank-Nisolcon method is obtained. If $\sigma = 0$, $s = 2$, $n = 1/2$, $k = j$ and $m = 0$, or if $\sigma = 1$, $s = 2$, $n = 1$, $k = j$ and $m = 1/2$, the explicit and implicit stages of the R11 method are obtained, respectively.

3   Convergence

When numerically solving a PDE, it is necessary that the numerical solution approaches the analytical solution (FORTUNA, 2012). To obtain the approximate solution, the number of operations depends on the Δx and Δt partitions. From a computational point of view, the lower the values of Δx and Δt, the closer the numerical solution will be to the analytical solution; however, the number of operations done by the computer rises. This approximation process can generate an accumulation of uncontrolled errors, characterizing a numerical method as stable or unstable.

To ensure that the numerical solution is as close as possible to the real solution of the problem, it is necessary and sufficient that the conditions established by Lax's theorem are satisfied (FORTUNA, 2012, CUMINATO; MENEGUETTE, 2013), that is:

Theorem 3.1. *A necessary and sufficient condition for the convergence of a numerical method, when applied to a well-posed initial value problem, is that the discretization scheme is consistent and stable.*

In this context, this section presents the theory of consistency and stability of numerical methods.

3. 1   Consistency

When the derivatives of a PDE are replaced by finite difference formulae, a finite difference equation (FDE) and an associated error are obtained, so that the PDE analytical solution does not satisfy the FDE precisely. This error is defined as Local Truncation Error (LTE) and appears due to the use of a finite amount of terms while expanding a function in Taylor series (FORTUNA, 2012, CUMINATO; MENEGUETTE, 2013).

To show that a FDE is consistent with PDE, consider the Δx, $\Delta t \to 0$ partitions. If LTE also tends to zero, then FDE is said to be consistent with PDE (FORTUNA, 2012, CUMINATO; MENEGUETTE, 2013).

3. 2 Stability

A numerical method is said to be stable when errors and disturbances in the solution are not amplified endlessly. One of the most used techniques in the study of the stability of finite difference equations is the Von Neumann criterion, which consists of expanding the Local Truncation Error E_i^n in Fourier series, as follows:

$$E_i^n = \sum_j \phi_j^n e^{IQx_i}, \tag{6}$$

where $I = \sqrt{-1}$ and for the $j-$th component, ϕ_j^n is its amplitude at time n and Q its wave number (CUMINATO; MENEGUETTE, 2013).

4 Convergence Analysis

In this section, the consistency and stability of the R11 method are checked. For that, Taylor series expansion is employed to show consistency and Von Neumann criterion to determine the stability region, as discussed in subsection 3. 1 and 3. 2, respectively. For the other methods, Explicit Euler and Crank-Nicolson, the convergence analysis is reviewed in Fortuna (2012), and Cuminato and Meneguette (2013).

4. 1 Consistency of the explicit stage of the R11 method

The consistency analysis presented in this subsection is performed for the explicit stage of the R11 method. For the implicit stage, the analysis is analogous and will not be presented.

Without loss of generality, consider in the equation (1) the thermal diffusivity term $\alpha(x,t)$ as constant and the source term $r(x,t)$ equal to zero, that is:

$$u_t = \alpha u_{xx}. \tag{7}$$

The explicit stage of the R11 method is given by:

$$U_i^{j+\frac{1}{2}} = (1 - \gamma)U_i^j + \frac{\gamma}{2}\left(U_{i-1}^j + U_{i+1}^j\right), \tag{8}$$

with $\gamma = \frac{\alpha\Delta t}{(\Delta x)^2}$.

Expanding in Taylor series the spatial component of U_{i+1}^j and U_{i-1}^j, given in (8), we obtain:

$$U_{i\pm 1}^j = U_i^j \pm \Delta x \frac{\partial}{\partial x}U_i^j \pm \frac{\Delta x^2}{2!}\frac{\partial^2}{\partial x^2}U_i^j \pm \frac{\Delta x^3}{3!}\frac{\partial^3}{\partial x^3}U_i^j \pm \mathcal{O}(\Delta x^4), \tag{9}$$

so that $\mathcal{O}(\Delta x^4)$, represents the upper and equal terms to the fourth order in the expansion in Taylor series.

Similarly, performing the Taylor series expansion in the time component of $U_i^{j+\frac{1}{2}}$, given in (8), it becomes:

$$U_i^{j+\frac{1}{2}} = U_i^j + \frac{\Delta t}{4}\frac{\partial}{\partial t}U_i^j + \frac{\Delta t}{4}\frac{\partial}{\partial t}\left(U_i^j + \frac{\Delta t}{2}\frac{\partial}{\partial t}U_i^j\right) + \mathcal{O}\left(\frac{\Delta t}{2}\right)^3, \tag{10}$$

so that $\mathcal{O}\left(\frac{\Delta t}{2}\right)^3$ represents the upper and equal terms of the third order in the expansion in Taylor series.

Yet, the expansion (10) deserves further consideration. Note that the term $U_i^{j+\frac{1}{2}}$ can be rewritten in the form

$$U_i^{j+\frac{1}{2}} \approx U_i^j + \frac{\Delta t}{2}\frac{\partial}{\partial t}U_i^j, \tag{11}$$

which comes from the discretization of the temporal term by the regressive finite difference formula, at time levels j e $j + \frac{1}{2}$. Thus, replacing (11) in (10) we get the approximate equation:

$$U_i^{j+\frac{1}{2}} = U_i^j + \frac{\Delta t}{4}\frac{\partial}{\partial t}U_i^j + \frac{\Delta t}{4}\frac{\partial}{\partial t}U_i^{j+\frac{1}{2}} + \mathcal{O}\left(\frac{\Delta t}{2}\right)^3. \tag{12}$$

On the other hand, applying the finite difference formula centred on the term $\frac{\partial}{\partial t}U_i^{j+\frac{1}{2}}$ of (12), encounter:

$$\frac{\partial}{\partial t}U_i^{j+\frac{1}{2}} \approx \frac{U_i^{j+1} - U_i^j}{\Delta t}. \tag{13}$$

In addition, the term $U_i^{j+\frac{1}{2}}$ is an arithmetic mean between the time levels j and $j + 1$, thus $U_i^{j+\frac{1}{2}} = \frac{U_i^{j+1} + U_i^j}{2}$, so $U_i^{j+1} = 2U_i^{j+\frac{1}{2}} - U_i^j$. Substituting the equality of U_i^{j+1} in (13), we have:

$$\frac{\partial}{\partial t}U_i^{j+\frac{1}{2}} \approx \frac{2U_i^{j+\frac{1}{2}} - 2U_i^j}{\Delta t}. \tag{14}$$

Replacing (14) in (12), the approximate equation follows:

$$U_i^{j+\frac{1}{2}} = U_i^j + \frac{\Delta t}{4}\frac{\partial}{\partial t}U_i^j + \frac{\Delta t}{4}\left(\frac{2U_i^{j+\frac{1}{2}} - 2U_i^j}{\Delta t}\right) + \mathcal{O}\left(\frac{\Delta t}{2}\right)^3. \tag{15}$$

Grouping the common terms in (15), and simplifying equation:

$$U_i^{j+\frac{1}{2}} = U_i^j + \frac{\Delta t}{2}\frac{\partial}{\partial t}U_i^j + 2\mathcal{O}\left(\frac{\Delta t}{2}\right)^3. \tag{16}$$

Finally, replacing the expansions (9) and (16) in (8), as well as some algebraic manipulations, we obtain:

$$\underbrace{\frac{\partial}{\partial t}U_i^j - \alpha\frac{\partial^2}{\partial x^2}U_i^j}_{\text{PDE}} = \underbrace{-4\mathcal{O}\left(\frac{\Delta t}{2}\right)^2 + 2\alpha\mathcal{O}(\Delta x)^2}_{\text{LTE}}. \tag{17}$$

Making Δx, $\Delta t \to 0$ in (17), the LTE tends to zero, leaving only the finite difference equation applied to a point in the mesh, that is

$$\frac{\partial}{\partial t}U_i^j = \alpha\frac{\partial^2}{\partial x^2}U_i^j. \tag{18}$$

Therefore, from (18), it is concluded that the first stage of the R11 method is consistent.

4. 2 **Stability of the R11 method**

To find the stability region of the R11 method, the Von Neumann criterion is used, as discussed in the subsection 3. 2.

Consider the explicit stage of the R11 method, given by

$$U_i^{j+\frac{1}{2}} = (1-\gamma)U_i^j + \frac{\gamma}{2}\left(U_{i-1}^j + U_{i+1}^j\right), \tag{19}$$

with $\gamma = \dfrac{\alpha\Delta t}{(\Delta x)^2}$.

Initially, each term of (19) is replaced by its equivalent Fourier series, like (6), that is

$$U_i^{j+\frac{1}{2}} = \phi^{j+\frac{1}{2}}e^{IQx_i}, \tag{20}$$

$$U_i^j = \phi^j e^{IQx_i}, \tag{21}$$

$$U_{i\pm 1}^j = \phi^j e^{IQ(x_i\pm\Delta x)} = \phi^j e^{IQx_i}e^{\pm IQ\Delta x}. \tag{22}$$

Substituting (20)-(22) in (19), we have

$$\phi^{j+\frac{1}{2}}e^{IQx_i} = (1-\gamma)\,\phi^j e^{IQx_i} + \frac{\gamma}{2}\left(\phi^j e^{IQx_i}e^{-IQ\Delta x} + \phi^j e^{IQx_i}e^{IQ\Delta x}\right). \tag{23}$$

As $e^{IQ\Delta x} + e^{-IQ\Delta x} = 2\cos(Q\Delta x)$, performing some algebraic manipulations in (23), it follows that:

$$\frac{\phi^{j+\frac{1}{2}}}{\phi^j} = 1 + \gamma\bigl(\cos(Q\Delta x) - 1\bigr). \tag{24}$$

Establishing the criterion of stability in (24):

$$\left|\frac{\phi^{j+\frac{1}{2}}}{\phi^j}\right| = \left|1 + \gamma\bigl(\cos(Q\Delta x) - 1\bigr)\right| \le 1, \tag{25}$$

that is,

$$1 + \gamma\bigl(\cos(Q\Delta x) - 1\bigr) \le 1, \tag{26}$$

$$1 + \gamma\bigl(\cos(Q\Delta x) - 1\bigr) \ge -1. \tag{27}$$

The inequality (26) is always satisfied, because when considering the maximum and minimum points of $\cos(Q\Delta x)$, we find $\gamma > 0$. On the other hand, from the second inequality (27), assuming the maximum and minimum points of $\cos(Q\Delta x)$, we find $\gamma \le 1$. Thus, a condition is initially found in the explicit stage is $0 < \gamma \le 1$.

Subsequently, we must analyze the stability conditions for the implicit stage of the R11 method, given by:

$$U_i^{j+1} = U_i^{j+\frac{1}{2}} + \frac{\gamma}{2}\left(U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}\right), \tag{28}$$

with $\gamma = \dfrac{\alpha\Delta t}{(\Delta x)^2}$.

Replacing each term of the implicit stage of the R11 method by its equivalent Fourier series, as (6), we obtain:

$$U_i^{j+\frac{1}{2}} = \phi^{j+\frac{1}{2}}e^{IQx_i}, \tag{29}$$

$$U_i^{j+1} = \phi^{j+1}e^{IQx_i}, \tag{30}$$

$$U_{i\pm1}^{j+1} = \phi^{j+1}e^{IQ(x_i\pm\Delta x)} = \phi^{j+1}e^{IQx_i}e^{\pm IQ\Delta x}. \tag{31}$$

Replacing (29)-(31) in (28), it follows that:

$$\phi^{j+1}e^{IQx_i} = \phi^{j+\frac{1}{2}}e^{IQx_i} + \frac{\gamma}{2}\left(\phi^{j+1}e^{IQx_i}e^{-IQ\Delta_x} - 2\phi^{j+1}e^{IQx_i} + \phi^{j+1}e^{IQx_i}e^{IQ\Delta_x} \right). \tag{32}$$

As $e^{IQ\Delta x} + e^{-IQ\Delta x} = 2\cos(Q\Delta x)$, while some algebraic manipulations in (32), we obtain:

$$\frac{\phi^{j+\frac{1}{2}}}{\phi^{j+1}} = 1 - \gamma\left(\cos\left(Q\Delta_x\right) - 1 \right). \tag{33}$$

Establishing the stability criterion at (33):

$$\left| \frac{\phi^{j+\frac{1}{2}}}{\phi^{j+1}} \right| = \left| 1 - \gamma\left(\cos\left(Q\Delta_x\right) - 1 \right) \right| \le 1, \tag{34}$$

that is,

$$1 - \gamma\left(\cos\left(Q\Delta_x\right) - 1 \right) \le 1, \tag{35}$$

$$1 - \gamma\left(\cos\left(Q\Delta_x\right) - 1 \right) \ge -1. \tag{36}$$

The inequalities (35) and (36) are always true when analyzing the maximum and minimum points of $\cos(Q\Delta x)$. Thus, for the R11 method to be a stable method, one must consider the conditions imposed in the explicit stage. Therefore, according to Lax theorem, the R11 method is convergent, as long as the condition $0 < \dfrac{\alpha\Delta t}{(\Delta x)^2} \le 1$ is respected.

Finally, the stability conditions to Explicit Euler, Crank-Nicolson and R11 methods are evaluated by following expressions:

$$\Phi_{\text{Exp-Euler}} = 1 + 2\gamma(cos(Q\Delta x) - 1) \qquad \text{(Explicit Euler)} \tag{37}$$

$$\Phi_{\text{CN}} = \frac{1 + \gamma(cos(Q\Delta x) - 1)}{1 - \gamma(cos(Q\Delta x) - 1)} \qquad \text{(Crank-Nicolson)} \tag{38}$$

$$\Phi_{\text{R11}} = 1 + \gamma(cos(Q\Delta x) - 1) \qquad \text{(R11)} \tag{39}$$

Table 1 shows, the stability criteria of the Explicit Euler, Crank-Nicolson and R11 methods, as well as their convergence orders.

As indicated in Table 1 the Explicit Euler and R11 methods are conditionally stable, while the Crank-Nicolson method is unconditionally stable. Figure 1 illustrates the stability region of these methods in the interval $r = [0.25, 0.75]$.

Figure 1 – Stability region: (a) Explicit Euler method; (b) Crank-Nicolson method; (c) R11 method.

(a)

(b)

(c)
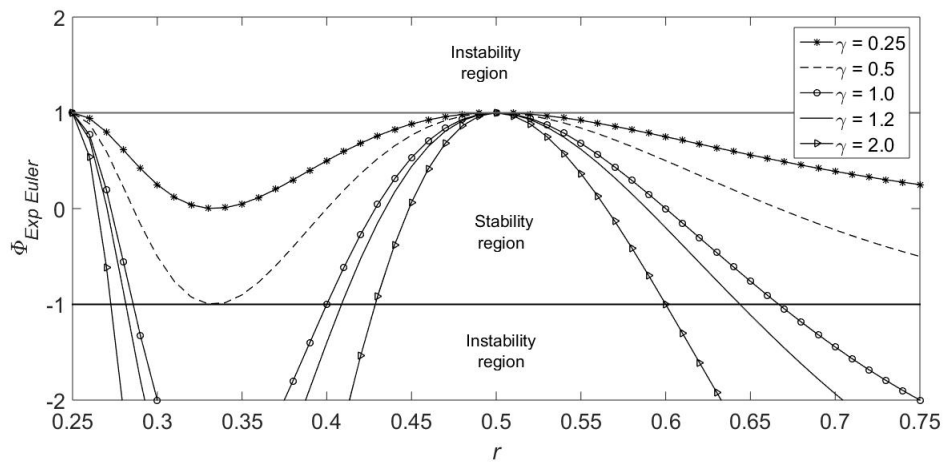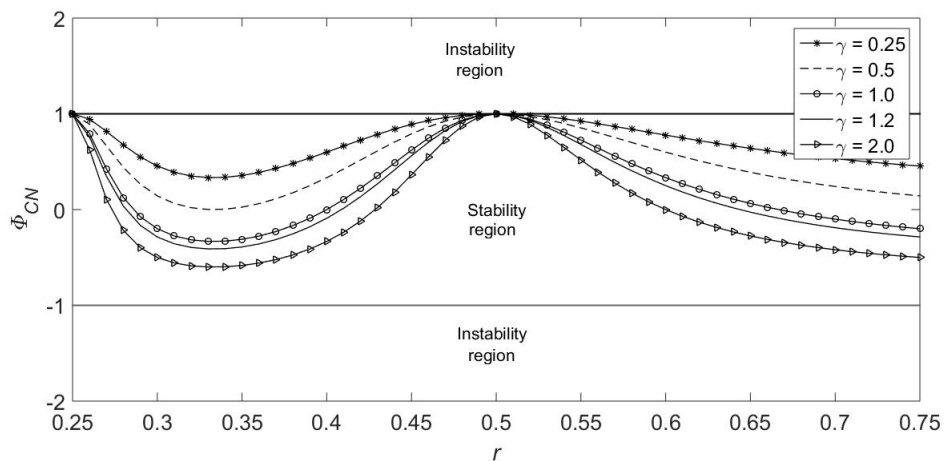
Source: The authors.

Table 1 – Stability interval and order of convergence of the methods.

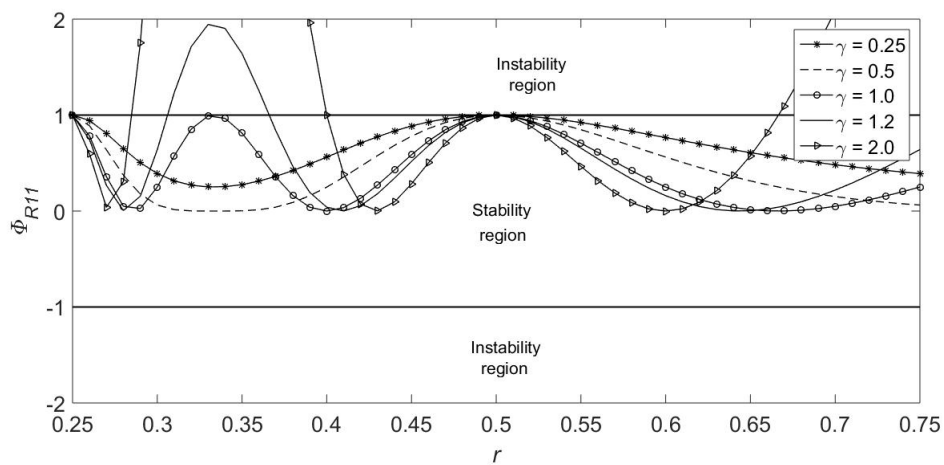| R11 method		Crank-Nicolson method		Explicit Euler method	
Stability	Convergence order	Stability	Convergence order	Stability	Convergence order
$\gamma \in (0, 1]$	$\mathcal{O}(\Delta t/2)^2$ $\mathcal{O}(\Delta x)^2$	$\gamma \in (0, \infty)$	$\mathcal{O}(\Delta t)^2$ $\mathcal{O}(\Delta x)^2$	$\gamma \in (0, 0.5]$	$\mathcal{O}(\Delta t)$ $\mathcal{O}(\Delta x)^2$

Source: The authors.

5   Results

All the results presented in this section come from the application of the numerical methods previously presented in model (1)-(3), for different initial and boundary conditions, in which the numerical solutions will be compared with the analytical solutions of the following examples. The spatial and temporal convergence orders of the methods used will also be presented.

5. 1   Exemple 1

Considering the domain $0 \leq x \leq 1$ and $0 \leq t \leq 1$, the one-dimensional model of heat diffusion (1)-(3), where the initial condition

$$u(x, 0) = 100sen(\pi x), \tag{40}$$

and the boundary conditions of the Dirichlet type

$$u(0, t) = 0,$$
$$u(1, t) = 0, \tag{41}$$

have the analytical solution given by (CUMINATO; MENEGUETTE, 2013):

$$u(x, t) = 100e^{-\pi^2 t}sen(\pi x). \tag{42}$$

Thus, given the equation (1), and using the initial and boundary conditions, equations (40)-(41), Figure 2 presents the solutions obtained by the numerical methods: Euler Explicit, Crank-Nicolson and R11, along with the analytical solution, when $M_x = 40$, $M_t = 3200$ and $\gamma = 0.5$.

From Figure 2(a)-(d), it is observed that the numerical solutions present, qualitatively, the same behavior of the analytical solution. As a way of assessing how close the numerical solutions are to the analytical solution, Figure 3 shows the behavior of the approximation errors of the numerical methods.

Figure 2 – Analytical and numerical solutions: (a) Analytical; (b) Explicit Euler; (c) Crank-Nicolson; (d) R11 method.
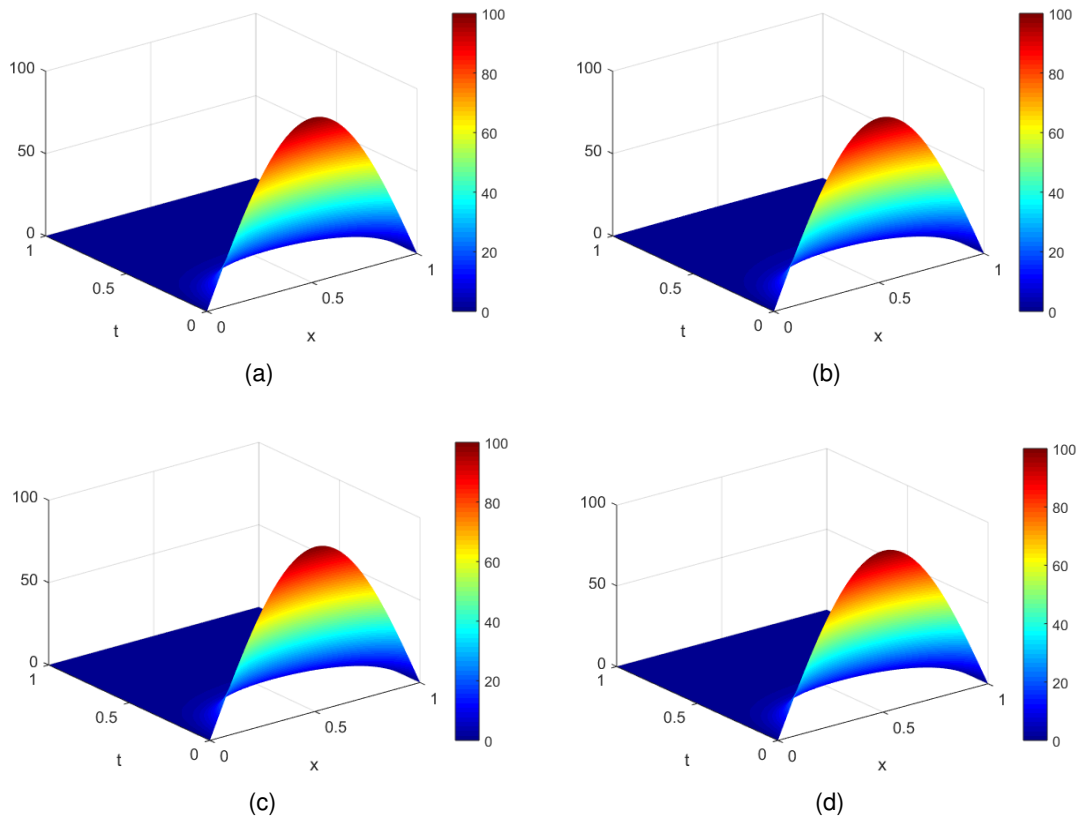
Source: The authors.

Figure 3 – Approximation errors of numerical methods in the problem domain: (a) Explicit Euler method; (b) Crank-Nicolson method; (c) R11 method.
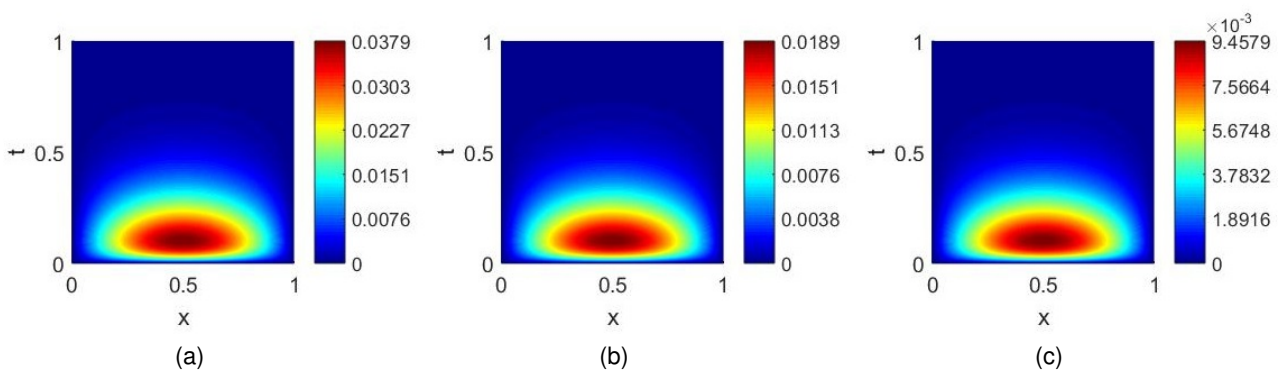
Source: The authors.

From the results presented in Figure 3, It is observed that the numerical methods Euler Explicit, Figure 3(a), Crank-Nicolson, Figure 3(b) and R11, Figure 3(c), reached the following maximum approximation error values: 3.7864×10^{-2}, 1.8882×10^{-2} and 9.4579×10^{-3}, respectively.

The R11 method had the lowest error values, while the Explicit Euler method had the highest values. Note that the results presented were obtained for $\gamma = 0.5$, that is, at the limit of the stability region of the Explicit Euler method (FORTUNA, 2012, CUMINATO; MENEGUETTE, 2013).

For a more detailed assessment of the characteristics of the methods employed, we set $M_x = 10$ and vary M_t, so that γ is within the stability region. The results are presented in Table 2 for the point $x = 0.5$, at the time level $t = 1$. The relative error E_{ij} for the calculation of the temporal convergence order, denoted by p_t, and the processing time of the solutions are also displayed.

Table 2 – Numerical solutions for $M_x = 10$, in $x = 0.5$ and $t = 1$ for several M_t values. The analytical solution is $u(0.5, 1) = 5.1723 \times 10^{-3}$.

γ	M_t	U_{ij}	E_{ij}	p_t	*cpu (s)*
	Explicit Euler method				
0.5	200	$4.3779{\times}10^{-3}$	$4.6106{\times}10^{-1}$	0.90194	0.11427
0.25	400	$4.9653{\times}10^{-3}$	$4.0020{\times}10^{-2}$	0.95192	0.10055
0.125	800	$5.2796{\times}10^{-3}$	$2.0745{\times}10^{-2}$	0.97619	0.13868
0.0625	1600	$5.4421{\times}10^{-3}$	$5.2162{\times}10^{-2}$	0.98816	0.15446
0.03125	3200	$5.5247{\times}10^{-3}$	$6.8132{\times}10^{-2}$		0.21561
0.015625	6400	$5.5663{\times}10^{-3}$	$7.6175{\times}10^{-2}$		0.34460
	R11 method				
1	100	$4.3779{\times}10^{-3}$	$1.5359{\times}10^{-1}$	1.8332	0.63517
0.5	200	$4.9653{\times}10^{-3}$	$4.0020{\times}10^{-2}$	1.9184	0.30289
0.25	400	$5.2796{\times}10^{-3}$	$2.0745{\times}10^{-2}$	1.9596	0.47587
0.125	800	$5.4421{\times}10^{-3}$	$5.2162{\times}10^{-2}$	1.9799	1.45370
0.0625	1600	$5.5247{\times}10^{-3}$	$6.8132{\times}10^{-2}$	1.9899	20.3880
0.03125	3200	$5.5663{\times}10^{-3}$	$7.6180{\times}10^{-2}$		37.0212
0.015625	6400	$5.5872{\times}10^{-3}$	$8.0215{\times}10^{-2}$		71.0903
	Crank-Nicolson method				
2	50	$5.4345{\times}10^{-3}$	$5.0693{\times}10^{-2}$	1.9866	0.11617
1	100	$5.5645{\times}10^{-3}$	$7.5827{\times}10^{-2}$	1.9967	0.10966
0.5	200	$5.5972{\times}10^{-3}$	$8.2149{\times}10^{-2}$	1.9992	0.33389
0.25	400	$5.6055{\times}10^{-3}$	$8.3754{\times}10^{-2}$	1.9998	0.21327
0.125	800	$5.6075{\times}10^{-3}$	$8.4141{\times}10^{-2}$	1.9999	0.20511
0.0625	1600	$5.6081{\times}10^{-3}$	$8.4257{\times}10^{-2}$	2.0000	0.30775
0.03125	3200	$5.6081{\times}10^{-3}$	$8.4257{\times}10^{-2}$		0.76860
0.015625	6400	$5.6082{\times}10^{-3}$	$9.8544{\times}10^{-2}$		1.76060

Source: The authors.

Similarly, $M_t = 25600$ is fixed and M_x is variable. The results are presented in Table 3, as well as the relative error for the calculation of the spatial convergence order, denoted by p_x, and the processing time of the solutions.

Table 3 – Numerical solutions for $M_t = 25600$, in $x = 0.5$ and $t = 1$ for several M_x values. The analytical solution is $u(0.5, 1) = 5.1723 \times 10^{-3}$.

| γ	M_x	U_{ij}	E_{ij}	p_x	*cpu (s)*
	Explicit Euler method				
0.0039	10	5.5977×10^{-3}	1.5558×10^{-1}	2.0505	0.11427
0.0156	20	5.2682×10^{-3}	1.8541×10^{-2}	2.0126	0.10055
0.0625	40	5.1887×10^{-3}	3.1707×10^{-3}		0.13868
0.2500	80	5.1690×10^{-3}	6.3801×10^{-4}		0.15446
	R11 method				
0.0039	10	5.6029×10^{-3}	8.3256×10^{-2}	2.0499	594
0.0156	20	5.2733×10^{-3}	1.0639×10^{-2}	2.0073	1287
0.0625	40	5.1937×10^{-3}	4.1374×10^{-3}	2.0000	2530
0.2500	80	5.1739×10^{-3}	3.1719×10^{-4}		5018
1.0000	160	5.1690×10^{-3}	6.3801×10^{-4}		14849
	Crank-Nicolson method				
0.0039	10	5.6082×10^{-3}	8.4276×10^{-2}	2.0516	190
0.0156	20	5.2782×10^{-3}	2.0474×10^{-2}	2.0073	459
0.0625	40	5.1986×10^{-3}	5.0848×10^{-3}	2.0000	1419
0.2500	80	5.1788×10^{-3}	1.2567×10^{-3}		4464
1.0000	160	5.1735×10^{-3}	2.3201×10^{-4}		18753

Source: The authors.

The R11 method has a greater stability region than the Explicit Euler method. In this context, while the Explicit Euler method obtained the result of $U(0.5, 1) = 4.3779 \times 10^{-3}$ (see Table 2) for a temporal refinement of $M_t = 200$, the R11 method obtained the same result; however, for a minor temporal refinement, that is, $M_t = 100$. The Crank-Nicolson method, on the other hand, presented results close to the analytical solution, obtaining relative errors in the order of 10^{-2}, although the approximation error increases as time is refined. This propagation of errors, in general, is corrected when space is also refined, according to the data presented in Table 3.

Regarding the estimated order of convergence p_t and p_x, it appears that they are consistent with the order of analytical convergence as shown in Table 1, that is, both Crank-Nicolson and R11 method have a second-order in time and space, while the Explicit Euler method has a first-order in

time and the second-order in space. Also, in all analyses, the Explicit Euler method obtained the shortest computational time, since this method does not require the resolution of a system of linear equations.

The analyses presented above were performed for a single specific point in the mesh. Now, to analyze the influence of all points of the mesh in obtaining the results, the spatial domain is discretized to $M_x = 10, 20, 40$ and 80, which represent $\Delta x = 0.1, 0.05, 0.025, 0.0125$, respectively, and the relative errors L_2 and L_∞ are calculated on the different meshes. The numerical results Figure 4 improve considerably when the number of elements in the mesh is increased.

Figure 4 – Errors in norms L_2 and L_∞: (a) Explicit Euler method; (b) Crank-Nicolson method; (c) R11 method.
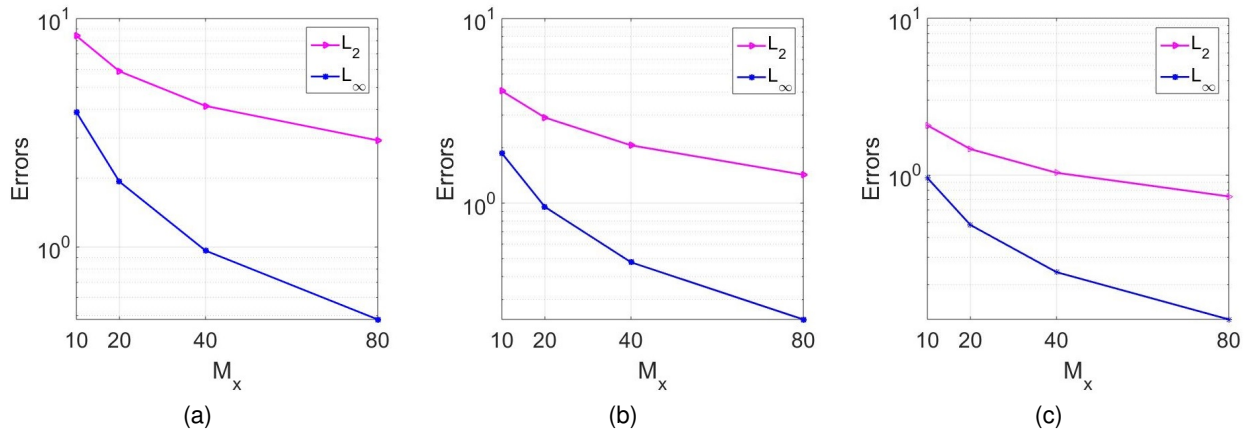
Source: The authors.

The R11 method produces numerical solutions with fewer error, as previously analyzed at point $x = 0.5$ in $t = 1$, with γ within its stability region. From the results presented in Figure 3, it is possible to indicate that this situation extends to all points of the mesh.

5. 2 **Exemple 2**

As shown in Example 1, the R11 method produces numerical solutions that are closer to the analytical solution. In this example, it will be analyzed for which values of γ this method produces fewer numerical errors. For this, the model (1) is considered with the initial condition

$$u(x,0) = \left(x - \frac{1}{3}\right)^6, \tag{43}$$

and the boundary conditions

$$u(0, t) = 0.0014\mathrm{e}^t,$$
$$u(1, t) = 0.0878\mathrm{e}^t. \tag{44}$$

The analytical solution to model (1), satisfying the conditions (43)-(44), is given by (GAO; SUN, 2012):

$$u(x, t) = \left(x - \frac{1}{3}\right)^6 \mathrm{e}^t. \tag{45}$$

Similarly, considering the domain $0 \leq x \leq 1$ e $0 \leq t \leq 1$, Figure 5(a)-(b) presents the analytical solution (45) and numerical solutions obtained by the R11 method, along with the approximation errors, Figure 5(c), for $M_x = 40$, $M_t = 3200$, and $\gamma = 0.5$.

Figure 5 – Solutions to one-dimensional heat diffusion equation (1) with initial and boundary conditions given by (43)-(44): (a) Analytical solution, equation (45); (b) R11 method; (c) Approximation errors of R11 method in the problem domain.

(a)

(b)

(c)
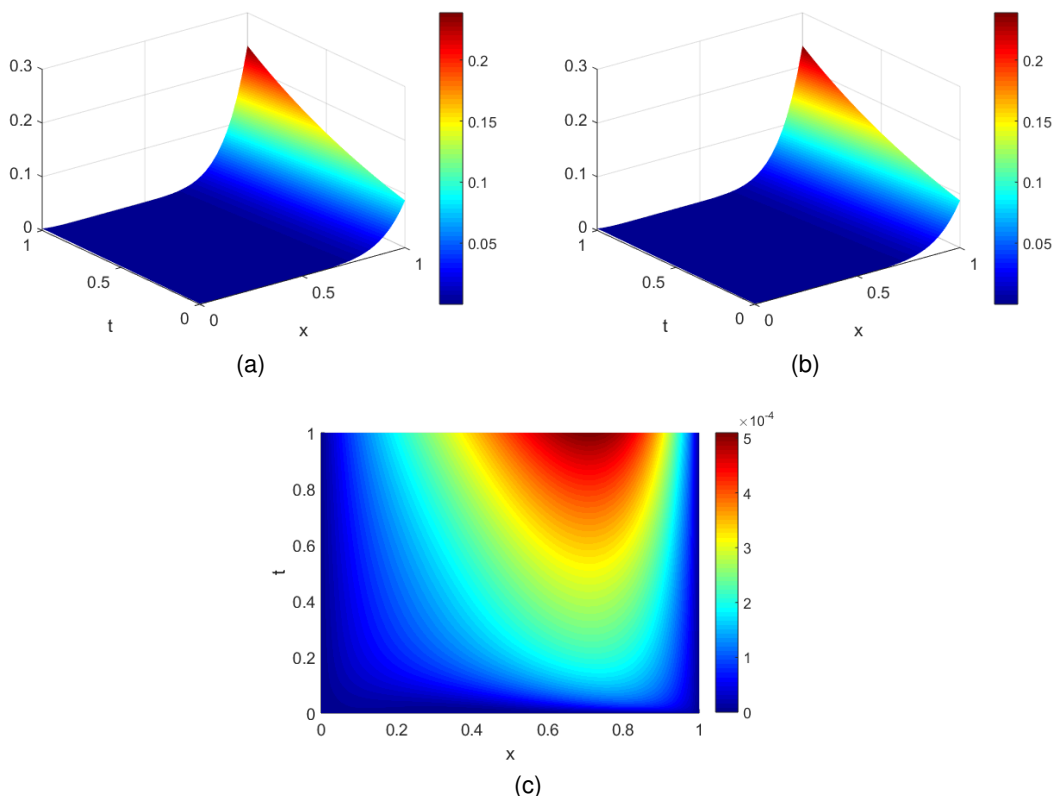
Source: The authors.

In regard to Figure 5, the R11 method produced similar results to the analytical solution's, with an absolute error in the order of 10^{-4}. To check for which values of γ the R11 method produces

fewer errors, we set $M_x = 20$ and vary M_t. The respective values of γ and the error at point $x = 0.5$ in $t = 1$ are shows in Figure 6.

Figure 6 – Behavior of the error produced by the R11 method, in the solution of the model (1), (43) and (44), for different values of γ.
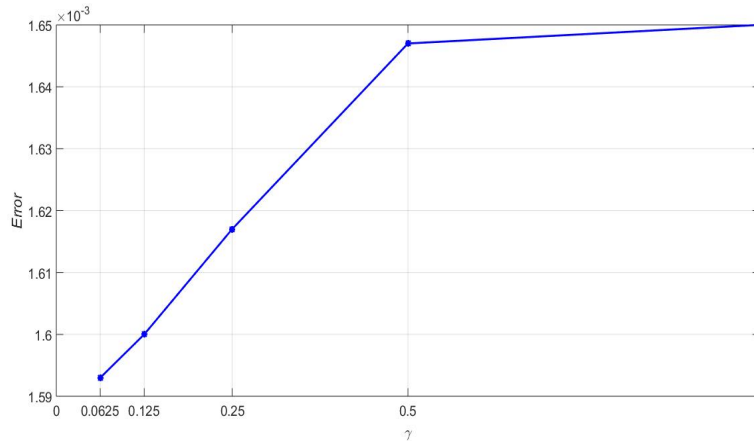
Source: The authors.

In Figure 6 the error is decreasing proportionally with γ values, that is, the lower the value of γ, fewer the error. On the other hand, when γ approaches the value that delimits its region of stability, the error of R11 method increase.

6 Conclusions

This article aimed to verify which of the methods: Euler Explicit, Crank-Nicolson and R11, produced results with lower numerical errors when used in the one-dimensional heat diffusion equation for different initial and boundary conditions.

In the examples presented, it was found that the R11 method was the numerical method that produced solutions with the least approximation error. When comparing this method with the others, it halves the number of calculations performed to obtain the numerical solution. This is due to the characteristic of the method that calculates the solution using two stages, that is, it uses time intervals $\Delta t/2$, therefore, the time refinement is less than the refinement of the other methods. On the other hand, the R11 method, as well as the Crank-Nicolson method, requires the iterative resolution of a system of linear equations. Thus, the processing time of its numerical solutions is longer when compared to the processing time of the Explicit Euler method.

The numerical approach conducted in this article did not require much computational effort in processing the solutions. In this context, it is concluded that, for the evaluated examples, the R11 method proved to be superior to the Explicit Euler and Crank-Nicolson methods.

Acknowledgments

References

ARAÚJO, J.; MÁRQUEZ, R. Simulação Numérica da Distribuição de Temperaturas em uma Barra Uniforme de Aço-Carbono com o Método de Crank-Nicolson. **Cadernos do IME - Série Matemática**, v. 6, n. 24, 2012. Available in: `https://www.e-publicacoes.uerj.br/index.php/cadmat/article/view/11895`. Access in: April 17, 2021.

BOYCE, W. E.; DIPRIMA, R. C. **Equações diferenciais elementares e problemas de valores de contorno**. Rio de Janeiro: Guanabara Dois, 1985.

CUMINATO, J. A.; MENEGUETTE, M. **Discretização de equações diferenciais parciais**: técnicas de diferenças finitas. Rio de Janeiro: Sociedade Brasileira de Matemática, 2013.

FARAGÓ, I.; PALENCIA, C. Sharpening the estimate of the stability constant in the maximum-norm of the Crank-Nicolson scheme for the one-dimensional heat equation. **Applied Numerical Mathematics**, v. 42, n. 1-3, p. 133-140, 2002. DOI: `https://doi.org/10.1016/S0168-9274(01)00146-5`.

FORTUNA, A. O. **Técnicas Computacionais para Dinâmica dos Fluídos**: Conceitos básicos e Aplicações. 2. ed. São Paulo: Editora da Universidade de São Paulo, 2012.

GAO, G. H.; SUN, Z. Z. Compact Difference Schemes for Heat Equation with Neumann Boundary Conditions (II). **Numerical Methods for Partial Differential Equations**. v. 29, n. 5, p. 1459–1486, 2012. DOI: `https://doi.org/10.1002/num.21760`.

GU, Y.; LEI, J.; FAN, C. M.; HE, X. Q. The generalized finite difference method for an inverse time-dependent source problem associated with three-dimensional heat equation. **Engineering Analysis with Boundary Elements**. v. 91, p. 74-81, 2018. DOI: `https://doi.org/10.1016/j.enganabound.2018.03.013`.

HAJIPOUR, M.; JAJARMI, A.; MALEK, A.; BALEANU, D. Positivity-preserving sixth-order implicit finite difference weighted essentially non-oscillatory scheme for the nonlinear heat equation. **Applied Mathematics and Computation**. v. 325, p. 146-158, 2018. DOI: `https://doi.org/10.1016/j.amc.2017.12.026`.

HORVÁTH, R. On the monotonicity conservation in numerical solutions of the heat equation. **Applied Numerical Mathematics**. v. 42, n. 1-3, p. 189-199, 2002. DOI: https://doi.org/10.1016/S0168-9274(01)00150-7.

KADALBAJOO, M. K.; AWASTHI, A. A numerical method based on Crank-Nicolson scheme for Burgers' equation. **Applied Mathematics and Computation**. v. 182, n. 2, p. 1430-1442, 2006. DOI: https://doi.org/10.1016/j.amc.2006.05.030.

KAZEM, S.; DEHGHAN, M. Application of finite difference method of lines on the heat equation. **Numerical Methods for Partial Differential Equations**. v. 34, n. 2, p. 626-660, 2018. DOI: https://doi.org/10.1002/num.22218.

LADEIA, C. A.; ROMEIRO, N. M. L.; NATTI, P. L.; CIRILO, E. R. Formulações Semi-Discretas para a Equação 1D de Burgers. **Tendências em Matemática Aplicada e Computacional**. v. 14, n. 3, p. 319-331, 2013. DOI: http://dx.doi.org/10.5540/tema.2013.014.03.0319.

PEREIRA, A J.; LISBOA, N. da H.; DIAS FILHO, J. H. Análise da estabilidade do método explícito para discretização de equações diferenciais parabólicas por meio de diferenças finitas. **C.Q.D.: Revista Eletrônica Paulista de Matemática**, Bauru, v. 11, p. 1-10, 2017. DOI: http://dx.doi.org/10.21167/cqdvol11ic201723169664ajpnhljhdf0110.

SAITA, T. M.; NATTI, P. L.; CIRILO, E. R.; ROMEIRO, N. M. L.; CANDEZANO, M. A. C.; ACUNA, R. A. B.; MORENO, L. C. G. Simulação numérica da dinâmica de coliformes fecais no lago Luruaco, Colômbia. **Tendências em Matemática Aplicada e Computacional**. v. 18, n. 3, p. 435-447, 2018. DOI: https://doi.org/10.5540/tema.2017.018.03.435.

WANG, Y. B.; NAKAGAWA, C. J.; YAMAMOTO, M. A numerical method for solving the inverse heat conduction problem without initial value. **Inverse Problems in Science and Engineering**. v. 18, n. 5, p. 655-671, 2010. DOI: https://doi.org/10.1080/17415971003698615.