Holow 1-D cylinders with heat generation: A sufficiently general approach to solving problems with a time variable Dirichlet condition

Cilindros Ocos 1-D com geração de calor: uma aproximação suficientemente geral de solução para problemas com condição de Dirichlet variável no tempo

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Abstract

In this work, a sufficiently general approximate solution to transient heat conduction problems in 1-D cylindrical geometry, with heat generation and a time variable Dirichlet conditions was presented using the Green function method. An important integral involving Bessel functions that is part of the solution has been solved in detail here. The results obtained with the use of this solution when applied in some particular cases of practical interest, were in good agreement with the solutions reported by the literature. We have adopted a methodology that consists of addressing a non-homogeneous problem solution with non-homogeneous boundary conditions in a non-homogeneous problem solution with homogeneous border conditions more two stationary solutions related to the given Dirichlet conditions. As a result, the solution obtained has no convergence problems at the boundaries of the cylindrical region with the temperature prescribed conditions.

Palavras-chave
Cilindro Oco
Funções de Bessel
Método de Funções de Green
Solução Geral Aproximada

Resumo

Neste trabalho, uma solução aproximada suficientemente geral para problemas de condução de calor transiente em geometria cilíndrica 1-D, com geração de calor e condições de Dirichlet variável no tempo foi apresentada usando o método de funções de Green. Uma importante integral envolvendo funções de Bessel, que faz parte da solução, foi aqui resolvida com detalhes. Os resultados obtidos com o uso dessa solução, quando aplicada em alguns casos particulares de interesse prático, ficaram em boa concordância com as soluções reportadas na literatura. Foi adotada uma metodologia que consiste em fatiar a solução do problema não homogêneo com condições de fronteira não homogêneas em uma solução do problema não homogêneo com condições de fronteira homogêneas mais duas soluções estacionárias relacionadas com as condições de Dirichlet dadas. Com isso, a solução obtida não tem problemas de convergência nas fronteiras da região cilíndrica com as condições prescritas de temperaturas.

1. Introduction

The problem of transient heat conduction in a long and thin 1-D hollow cylinder of homogeneous and isotropic material, with heat generation $g(r,t)$ per unit of time, per unit volume $\left(\frac{W}{m^3}\right)$ and with variable Dirichlet conditions, which according to Hahn and Özisik (2012, p. 329) is given by

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{g(r,t)}{k}=\frac{1}{\kappa}\frac{\partial T}{\partial t}, \quad a<r<b, \quad t>0 \ , \tag{1}$$

where $T=T(r,t)$ is the temperature in the cylindrical region, k is the thermal conductivity of the material, while κ is the thermal diffusivity of the substance, that is, it represents the thermal-physical properties of the medium (HAHN; ÖZISIK, 2012). Together with equation (1) there are imposed Dirichlet boundary conditions, $T_S = f(t)$ and a given initial condition which are represented by the equations:

$$\begin{cases} CF1: \quad T(a,t)=f_1(t), \quad t>0, \\ CF2: \quad T(b,t)=f_2(t), \quad t>0, \end{cases} \tag{2}$$

$$CI: \quad T(r,0)=F(r), \quad a\le r\le b. \tag{3}$$

Hollow cylinders have very practical importance, especially in engineering. In mechanics, for example, hydraulic hollow piston cylinders are widely used. In electricity, these cylinders are made of ferrite discs, which is a material formed by iron oxide with magnetic properties, and are commonly used to avoid high temperature variations and heat peaks in electrical currents that pass through cables, like a laptop power cord (ROMER, 2013). These cylinders are also used as cladding in cylindrical layers where heat is generated, for example, in nuclear and thermoelectric plants. According to Rodrigues and Mesquita (2017), uranium-zirconium hydride fuel elements of diameter $D=37,7mm$ encapsulated in aluminum type AL11OF of thickness $e=0,76\,mm$ are used in the Triga IPR-R1 research reactor, located at the UFMG Campus.

These aluminum alloys are widely adopted as fuel cladding in uranium (U), plutonium (Pu) or thorium (Th) based fuel element reactors due to their high thermal conductivity, low thermal neutron capture section, good strength to corrosion, availability and low cost (PERROTTA, 1999). It is known that a one-dimensional analysis is generally not realistic for a 3-D problem, and even two-dimensional movements of a fluid, depending dependent on time, do not actually exist (MEYER, 2007). However, Fox, Pritchard and MacDonald (2015) state that in many engineering problems, even a 1-D analysis is adequate to provide approximate solutions within the desired precision. In this sense, infinite circular cylinders, that is, long, $\frac{L}{r_0}\ge 10$, and thin, are suitable for a one-dimensional analysis because they meet the requirement of radial symmetry, $\partial_r T\left(0,t\right)=0$.

In this way, many problems of practical importance can be approached by the mathematical formulation described by eqs (1-3). If boundary conditions are functions of time, the variable separation method must be discarded (FERNANDES, 2009). Although the Laplace transform has been widely used by Carslaw and Jaeger (2011) to obtain a general solution for homogeneous, transient heat conduction problems in 1-D hollow cylinders, it does not contemplate Dirichlet conditions more general than constant values. Furthermore, the initial temperature over the cylindrical region must be zero.

Cinelli (1965) presented an article in which he used the finite Hankel transform method to solve heat conduction problems in this geometry, in order to study with more emphasis, the Cauchy boundary conditions $\partial_S T = h\left(T_S - T_\infty\right)$, on the faces of the region, which occurs, for example, in nuclear reactors. Cauchy conditions are easier to perform from a physical point of view, for example a ventilation system that moves a fluid such as coolant air from one point to another, whereas a Dirichlet condition can represent a change in phase of a substance that occurs on the surface, such as evaporation/cooking (HAHN; ÖZISIK, 2012). According to Hahn and Özisik (2012) the most powerful method to solve transient and non-homogeneous heat conduction problems is the approximation by Green functions. The great strength of the use of these functions lies in the possibility of obtaining solutions, of this type, to more varied and complex problems, including inhomogeneities that vary in time and space (FERNANDES, 2009).

To obtain Green's function it is necessary to solve the associated homogeneous problem by separating variables. Once the Green's function is obtained for a specific problem, the analytical solution of temperature distribution is immediately available. This solution involves several terms and each of them has a physical meaning (HAHN; ÖZISIK, 2012). However, when temporal Dirichlet conditions such as those specified by eqs (2) and (3) are used, the solution obtained using Green's functions may not converge uniformly in the regions close to the boundary points as can be seen in example 8.8 of Hahn and Özisik (2012, p. 329).

In order to remove or even mitigate this difficulty, Hahn and Özisik (2012) suggest slicing the original solution, then the non-homogeneous problem with non-homogeneous boundary conditions is transformed into a non-homogeneous problem, but now with homogeneous boundary conditions . Our objective is to use the Green function method to obtain a sufficiently general expression, which is an approximation of the solution to transient heat conduction problems, with heat generation and with non-homogeneous Dirichlet boundary conditions changing over time, and a given initial condition in the 1-D geometry on thin and long hollow cylinders, that is, heat problems that can be represented by eqs (1-4). An important integral involving Bessel functions that is part of the approximation of the established general solution can be solved, which produced simplifications in this equation when particular cases of it were analyzed. Some of these special cases reported in the existing literature were used for validation by comparing the general solution when applied to each case analyzed.

2. Materials and methods

The homogeneous heat conduction equation in transient regime, and in a 1-D system in cylindrical coordinates is given by the equation (CARSLAW; JAEGER, 2011):

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}=\frac{1}{\kappa}\frac{\partial T}{\partial t}. \tag{4}$$

2.1. Sturm-Liouville problem in cylindrical coordinates

An equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[q(x)+\lambda w(x)]\,y=0\,, \tag{5}$$

which satisfies certain boundary conditions on an interval $a \le x \le b$, where $p(x),\ q(x)$ and $w(x)$ are continuous functions on that interval, with $p(x)$ differentiable, is called the Sturm-Liouville equation. The search for non-trivial solutions for this equation with boundary conditions at the extremes of the interval is called the Sturm-Liouville problem (DAVIS, 1963, SOTOMAYOR, 1979). Such solutions are called eigenfunctions, while the corresponding λ´s are called eigenvalues. This problem in cylindrical coordinates is given as in eq. (5) replacing the variable x with $r,$ that is,

$$\frac{d}{dr}\left[p(r)\frac{dy}{dr}\right]+[q(r)+\lambda w(r)]y=0. \tag{6}$$

In particular, if $p(r)=r$, $q(r)=-\frac{v}{r}$ and $w(r)=r$, and making $y=R$, we have the ordinary differential equation in the variable r given by

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\lambda^2-\frac{v^2}{r^2}\right)R=0\,, \tag{7}$$

where $r>0$ and generally $v=0,1,2...$

Equation (7) subject to the homogeneous boundary conditions given by

$$\begin{aligned} A_1\frac{dR}{dr}+A_2R=0, &\quad r=a,\\ B_1\frac{dR}{dr}+B_2R=0, &\quad r=b, \end{aligned} \tag{8}$$

where A_1, A_2, B_1, B_2 are non-zero constants, is a Sturm-Liouville problem in cylindrical (or spherical) coordinates. The eigenfunctions $R_v(r,\lambda_n)$, under the condition of $\lambda>0$ (HAHN; ÖZISIK, 2012) that satisfy equation (7) and the conditions given by equation (8) constitute an orthogonal set of functions in space $C^2([a,b])$ with respect to the inner product given by

$$\langle f,g\rangle=\int_a^b r\, f(r)\,g(r)dr\,, \tag{9}$$

and the quadratic norm of each eigenfunction given by (ÖZISIK; HAHN, 2012)

$$N\left(\lambda_n\right) = N\left[R_v(r,\lambda_n)\right] = \int_a^b r\, R_v^2(r,\lambda_n)dr\ . \tag{10}$$

According to Davis (1963) every R-integrable function in space $C^2\left([a,b]\right)$ that satisfies the boundary conditions given by equation (8) can be expanded in a series that converges point to point in this interval, that is,

$$R(r) = \sum_{n=1}^{\infty} C_n R_v\left(r,\lambda_n\right), \tag{11}$$

where $C_n = \dfrac{1}{N\left(\lambda_n\right)}\displaystyle\int_a^b r\, R(r)\, R_v(r,\lambda_n)\, dr, \quad n = 1, 2, \ldots$.

The equation (7) is called Bessel equation and its general solution is the equation given by (HAHN; ÖZISIK, 2012)

$$R(r) = c_1 J_v(\lambda r) + c_2 Y_v(\lambda r), \tag{12}$$

where $J_v(\lambda r)$ and $Y_v(\lambda r)$ are respectively Bessel functions of the first and second kind of order v.

2.2. Variables Separation Method

A standard method to obtain a solution of eq. (4) is to assume that variables are separable

$$T(r,t) = G(r)\, H(t), \tag{13}$$

where G and H are just functions of r and t, respectively. Substituting equation (13) into equation (4) results in two ordinary differential equations (ODEs) independent in form (HAHN; ÖZISIK, 2012)

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{d\,G}{dr} = \frac{1}{\kappa H}\frac{d\,H}{dt} = -\lambda^2, \tag{14}$$

that is,

$$\frac{1}{\kappa H}\frac{d\,H}{dt} = -\lambda^2, \tag{15}$$

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} + \lambda^2 G = 0\,. \tag{16}$$

The equation (15) can be solved directly by separating variables to obtain,

$$H(t) = c_1 e^{-\kappa\lambda^2 t}, \tag{17}$$

while the ODE in variable r is a Bessel equation of order $v = 0$, whose general solution is given by equation (12) in the form

$$G(r) = C_2 J_0\left(\lambda r\right) + C_3 Y_0\left(\lambda r\right), \tag{18}$$

where the constants can be eliminated through suitable initial conditions. Therefore, a general solution of equation (4) is given by the product of the individual solutions given by equations (17) and (18), that is,

$$T(r,t) = C_1 e^{-\kappa\lambda^2 t}\left[C_2 J_0(\lambda r) + C_3 Y_0(\lambda r)\right], \tag{19}$$

or omitting the constants,

$$T(r,t) = e^{-\kappa\lambda^2 t} G_0(r,\lambda). \tag{20}$$

As for each $\lambda_n > 0$, we can associate an eigenfunction $G_v(r,\lambda_n)$ and since equation (4) is linear, the most general solution will be obtained by the sum of all products of equations (17) and (18)

$$T(r,t) = \sum_{n=1}^{\infty} C_n e^{-\kappa\lambda_n^2 t} G_0(r,\lambda_n), \tag{21}$$

where the coefficients C_n are obtained from equation (11) when the initial condition $T(r,0) = F(r)$ is known.

3. Results and discussions

The homogeneous version of the problem given by equations (1-4) is in the form

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{1}{\kappa}\frac{\partial\psi}{\partial t}, \quad a < r < b, \quad t > 0, \tag{22}$$

$$CF1: \quad \psi(a,t) = 0, \quad t > 0, \tag{23}$$

$$CF2: \quad \psi(b,t) = 0, \quad t > 0, \tag{24}$$

$$CI: \quad \psi(r,\ 0) = F(r), \quad a \le r \le b. \tag{25}$$

Using the variable separation method described in section 2.2, equations (13) and (18) and the $CF2$, condition given by equation (24),

$$C_3 = -C_2 \frac{J_0(\lambda b)}{Y_0(\lambda b)}, \tag{26}$$

where $Y_0(\lambda b) \neq 0$, otherwise we would only have the trivial solution. From equations (18) and (26) results

$$G(r) = C_4\left[J_0(\lambda r)Y_0(\lambda b) - J_0(\lambda b)Y_0(\lambda r)\right] = C_4 G_0(\lambda r), \tag{27}$$

with $C_4 = \dfrac{C_2}{Y_0(\lambda b)}$. From equations (13) and (27) and from the $CF1$ condition given by equation (23) we have the transcendental equation

$$G_0(\lambda a) = 0, \tag{28}$$

where

$$G_0(\lambda r) = \left[J_0(\lambda r)\, Y_0(\lambda b) - J_0(\lambda b)\, Y_0(\lambda r) \right] \tag{29}$$

is an eigenfunction associated with the eigenvalue λ defined in the interval $a \le r \le b$.

The roots λ_n, $n = 1, 2, ...$ are all real and simple, where for each root λ there is a corresponding root $-\lambda$ (CARSLAW; JAEGER, 2011). From equations (13), (17) and (27) we have

$$\psi(r,t) = \sum_{n=1}^{\infty} C_n\, G_0(\lambda_n r)\, e^{-\kappa \lambda_n^2 t}, \quad a < r < b, \quad t > 0, \tag{30}$$

so that the coefficients are obtained as in equation (11) by substituting $R(r)$ for $F(r)$, *i.e.*,

$$C_n = \frac{1}{N(\lambda_n)} \int_a^b r\, G_0(\lambda_n r)\, F(r)\, dr\,, \tag{31}$$

being the inverse of the given norm using equation (10),

$$\frac{1}{N(\lambda_n)} = \frac{1}{\int_a^b r\, G_0^2(\lambda_n r)\, dr}\,. \tag{32}$$

Had we used condition $CF1$, given by equation (23) in equation (26), instead of $CF2$, we could use case 4 from Table 2.3 of Hahn and Özisik (2012, p. 54) to obtain $\frac{1}{N(\lambda_n)}$ directly. From equations (30) and (31) we have

$$\psi(r,t) = \sum_{n=1}^{\infty} \frac{G_0(\lambda_n r)}{N(\lambda_n)} e^{-\kappa \lambda_n^2 t} \int_a^b G_0(\lambda_n r')\, F(r')\, r' dr' = \int_a^b \sum_{n=1}^{\infty} \frac{G_0(\lambda_n r)}{N(\lambda_n)} e^{-\kappa \lambda_n^2 t} G_0(\lambda_n r')\, F(r')\, r' dr'\,. \tag{33}$$

According to equation (8-14) of Hahn and Özisik (2012, p. 306) the solution of the homogeneous problem given by equations (22-25) in terms of Green's functions can be put in the form

$$\psi(r,t) = \int_a^b G(r,t,r',\tau)_{\tau=0}\, F(r')\, r' dr'\,. \tag{34}$$

Therefore, by comparing equations (33) and (34) we can obtain the Green’s function developed in $\tau = 0$ for this problem in the form

$$G(r,t,r',\tau)_{\tau=0} = \sum_{n=1}^{\infty} \frac{1}{N(\lambda_n)} G_0(\lambda_n r)\; e^{-\kappa \lambda_n^2 t} G_0(\lambda_n r')\,, \tag{35}$$

or, in general,

$$G(r,t,r',\tau) = \sum_{n=1}^{\infty} \frac{1}{N(\lambda_n)} G_0(\lambda_n r)\, e^{-\kappa \lambda_n^2 (t-\tau)} G_0(\lambda_n r')\,, \tag{36}$$

which is the Green’s function for the non-homogeneous version given by equations (22-25).

Instead of using tables to obtain $\dfrac{1}{N(\lambda_n)}$, Carslaw and Jaeger (2011) succinctly proved that

$$N(\lambda)=\int_a^b rG_0^2(\lambda r)dr=\frac{1}{2\lambda^2}\left(r\frac{dG_0}{dr}\right)^2\Bigg|_a^b. \tag{37}$$

The equation (37) corresponds to the original equation (5) of those authors, in which we replaced $U_0(\alpha r)$ by $G_0(\lambda r)$. According to equations (7) and (8) by Carslaw and Jaeger (2011, p. 206), there is

$$\left(r\frac{dG_0(\lambda r)}{dr}\right)_{r=a}=-\frac{2}{\pi\rho} \quad \text{e} \quad \left(\frac{rdG_0(\lambda r)}{dr}\right)_{r=b}=-\frac{2}{\pi}, \tag{38}$$

where

$$\rho=\frac{J_0(\lambda a)}{J_0(\lambda b)}. \tag{39}$$

From equations (37-39) we have

$$N(\lambda_n)=2\frac{J_0^2(\lambda a)-J_0^2(\lambda b)}{\pi^2\lambda^2 J_0^2(\lambda a)}. \tag{40}$$

From equations (36) and (40) we can express Green's function as

$$G(r,t,r',\tau)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{\lambda_n^2 J_0^2(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2(t-\tau)}G_0(\lambda_n r)G_0(\lambda_n r'). \tag{41}$$

In particular, when the initial condition is constant, the integral that appears on the right side of equation (33) can be simplified. Carslaw and Jaeger (2011) presented the following proposition

$$\int_a^b rG_0(\lambda r)\,dr=-\frac{1}{\lambda^2}\left(r\frac{dG_0(\lambda r)}{dr}\right)\Bigg|_{\grave{a}}^b=\frac{2\left(J_0(\lambda a)-J_0(\lambda b)\right)}{\pi\,\lambda^2 J_0(\lambda a)}. \tag{42}$$

Here the negative sign in the middle term of this equation should be positive, because as the right side of equation (42) is, it has the opposite sign. In order to resolve this mistake, we will prove that

$$\int_a^b rG_0(\lambda r)\,dr=+\frac{1}{2\lambda^2}\left[\left(\frac{r\,dG_0(\lambda r)}{dr}\right)\right]_a^b=\frac{2\left(J_0(\lambda a)-J_0(\lambda b)\right)}{\pi\,\lambda^2 J_0(\lambda a)}. \tag{43}$$

From equation (29),

$$\begin{aligned}\int_a^b rG_0(\lambda r)\,dr&=Y_0(\lambda b)\int_a^b rJ_0(\lambda r)\,dr-J_0(\lambda b)\int_a^b rY_0(\lambda r)\,dr\\&=Y_0(\lambda b)\left(\frac{1}{\lambda}rJ_1(\lambda r)\Big|_a^b\right)-J_0(\lambda b)\left(\frac{1}{\lambda}rY_1(\lambda r)\Big|_a^b\right),\end{aligned} \tag{44}$$

where expressions in parentheses are taken from equation (22) of Hahn and Özisik (2012, p. 694). The equation (44) can then be placed in the form

$$\int_a^b rG_0(\lambda r)\,dr = \frac{b}{\lambda}\left\{Y_0(\lambda b)\left[\frac{dJ_0(\lambda b)}{dr}\right] - J_0(\lambda b)\left[\frac{dY_0(\lambda b)}{dr}\right]\right\} + \\ + \frac{a}{\lambda}\left\{J_0(\lambda b)\left[\frac{dY_0(\lambda a)}{dr}\right] - Y_0(\lambda b)\left[\frac{dJ_0(\lambda a)}{dr}\right]\right\}. \tag{45}$$

Using the equation (38) with the signals corrected to positive, we have from equations (29) and (45) that

$$\int_a^b rG_0(\lambda r)\,dr = \frac{b}{\lambda}\left[-\frac{1}{b\lambda}\right]\left[r\frac{dG_0(\lambda r)}{dr}\right]_{r=b} + \frac{a}{\lambda}\left[\frac{1}{a\lambda}\right]\left[r\frac{dG_0(\lambda r)}{dr}\right]_{r=a} = \frac{2\left(J_0(\lambda a) - J_0(\lambda b)\right)}{\pi\lambda^2 J_0(\lambda a)}. \tag{46}$$

The solution of the heat conduction problem given by equations (1-4) in terms of the Green's function given by equation (41) can be obtained using the equation (8-14) of Hahn and Özisik (2012, p. 306) in the form

$$T(r,t) = \int_{r'=a}^{b} G(r,t,r',\tau)_{\tau=0} F(r') r'\,dr' + \frac{\kappa}{k}\int_0^t\int_a^b G(r,t,r',\tau)\,g(r',\tau)\,r'\,dr'\,d\tau + \\ + \kappa\int_{r'=a}^{b}\left[r'\frac{\partial G}{\partial r'}(r,t,r',\tau)\right]_{r'=a} f_1(\tau)\,d\tau - \kappa\int_a^b\left[r'\frac{\partial G}{\partial r'}(r,t,r',\tau)\right]_{r'=b} f_2(\tau)\,d\tau \tag{47}$$

.

The problem with this general solution is that the last two integral terms that carry prescribed temperature information at the boundaries may not converge uniformly on $r = a$ or on $r = b$, for the values of $f_1(t)$ or $f_2(t)$, that is, non-homogeneous boundary conditions can lead to difficulties in the convergence of the solution close to them (ÖZISIK, 1993). An example of this difficulty in converging the solution can be seen in problem 8.8 whose solution is given by equation (8-138) of Hahn and Özisik (2012, p. 331). According to these authors, one of the alternatives to alleviate this difficulty is to slice the original problem using the methodology proposed by Özisik (1993) that will be adopted here, and briefly described. Such methodology transforms in some cases, in particular the one we are analyzing, heat conduction problems in transient regime with non-homogeneous boundary conditions, into non-homogeneous problems, but with homogeneous boundary conditions. It is clear that if the non-homogeneous boundary conditions are transformed into homogeneous by this technique, the solution given by equation (47) should no longer have the contributions of the integral terms as functions of $f_1(\tau)$ and $f_2(\tau)$.

It will be assumed that the solution of the problem given by equations (1-4) can be separated by the contribution of three components in the form

$$T(r,t) = \theta(r,t) + \varphi_1(r)\,f_1(t) + \varphi_2(r)\,f_2(t), \tag{48}$$

where $\varphi_1(r)$ and $\varphi_2(r)$ satisfy the following Cauchy problems given by ordinary differential equations in the form

$$\begin{aligned} & r\frac{d^2\varphi_1(r)}{dr^2}+\frac{d\varphi_1(r)}{dr}=0, \quad a<r<b, \\ & \varphi_1(a)=1, \quad \varphi_1(b)=0 \end{aligned} \tag{49}$$

and

$$\begin{aligned} & r\frac{d^2\varphi_2(r)}{dr^2}+\frac{d\varphi_2(r)}{dr}=0, \quad a<r<b, \\ & \varphi_2(a)=0, \quad \varphi_2(b)=1. \end{aligned} \tag{50}$$

Making $r\dfrac{d\varphi_i(r)}{dr}=B_i, \quad i=1,2$, we have $\varphi_i(r)=A_i+B_i\ln(r), \quad i=1,\ 2.$ Hence, and using the boundary conditions given by equations (49) and (50), it results that

$$\varphi_1(r)=\frac{\ln\left(\dfrac{r}{b}\right)}{\ln\left(\dfrac{a}{b}\right)} \quad \text{e} \quad \varphi_2(r)=\frac{\ln\left(\dfrac{r}{a}\right)}{\ln\left(\dfrac{b}{a}\right)}. \tag{51}$$

The functions $\varphi_i(r)$ are the stationary solutions to the problems of heat conduction in a 1-D hollow cylinder without heat generation given by equations (49-50). From equations (1-4), (48) and (51) results the problem $\theta=\theta(r,t)$ given by

$$\begin{aligned} & \frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\theta+g^*=\frac{1}{k}\frac{\partial\theta}{\partial t}, \quad a<r<b, \quad t>0 \\ & CF1: \quad \theta(a,t)=0, \quad t>0, \\ & CF2: \quad \theta(b,t)=0, \quad t>0, \\ & C.I: \quad \theta(r,0)=F^*(r), \quad a\le r\le b, \end{aligned} \tag{52}$$

where

$$g^*(r,t)=\frac{g(r,t)}{k}-\frac{1}{\kappa}\left[\varphi_1(r)\frac{d}{dt}f_1(t)+\varphi_2(r)\frac{d}{dt}f_2(t)\right] \tag{53}$$

and

$$F^*(r)=F(r)-\left[\varphi_1(r)f_1(0)+\varphi_2(r)f_2(0)\right]. \tag{54}$$

The equations (53) and (54) correspond to equations (1.50a) and (1.50b) of Özisik (1993, p. 23). This method can be extended to multidimensional problems, since the boundary conditions are only a function of time.

Green’s function given by equation (41) can be used to solve the problem given by equation (52) in the form

$$\theta(r,t)=\int_a^b G(r,t,r',\tau)_{\tau=0}F^*(r')r'dr'+\kappa\int_0^t\left(\int_a^b G(r,t,r',\tau)\,g^*(r',\tau)\,r'dr'\right)d\tau\,. \tag{55}$$

From equations (48) and (55) results the approximate solution by Green's functions for the problem given by equations (1-4) as

$$\begin{aligned}T(r,t)=&\int_a^b G(r,t,r',\tau)_{\tau=0}F^*(r')r'dr'+\kappa\int_0^t\left(\int_a^b G(r,t,r',\tau)\,g^*(r',\tau)\,r'dr'\right)d\tau+\\&+\varphi_1(r)\,f_1(t)+\varphi_2(r)\,f_2(t),\end{aligned} \tag{56}$$

where $G(r,t,r',\tau)$ is given as in equation (41) and $\varphi_i(r)$, $i=1,\ 2$ are given as in equation (51). Note that now the solution obtained by equation (56) converges to the values prescribed at the boundary by equations (2) and (3), that is,

$$T(a,t)=f_1(t)\ \text{and}\ T(b,t)=f_2(t),\quad t>0\,. \tag{57}$$

To see this, simply use equations (28) and (51) in $r=a$ and equations (29) and (51) in $r=b$. Note that the first term on the right side of equation (55) is the contribution of the initial condition $F^*(r)$ from $\theta(r,t)$ to $T(r,t)$, that is, Green's function is evolved in $\tau=0$, multiplied by $F^*(r)$ and then integrated in the $a\le r\le b$ domain. The second term brings the contribution of the generation term $g^*(r',\tau)$ from $\theta(r,t)$ to $T(r,t)$, that is, it is the Green's function multiplied by por $g^*(r',\tau)$ and then integrated in this space. The third term brings the contributions of the Dirichlet boundary conditions. From equations (41), (51) and (54), the equation (56) can then be rewritten as follows

$$\begin{aligned}T(r,t)=&\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{\lambda_n^2J_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)\int_a^b F^*(r')r'G_0(\lambda_n r')dr'+\\&+\frac{\kappa\pi^2}{2}\sum_{n=1}^{\infty}\frac{\lambda_n^2J_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)\int_0^t\int_a^b e^{-\kappa\lambda_n^2\tau}G_0(\lambda_n r')\,g^*(r',\tau)r'dr'd\tau+\\&+\frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)}f_1(t)+\frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)}f_2(t),\end{aligned} \tag{58}$$

where the equations (53) and (54) are again presented in the following form

$$\begin{aligned}g^*(r,t)&=\frac{g(r,t)}{k}-\frac{1}{\kappa}\left[\varphi_1(r)\frac{d}{dt}f_1(t)+\varphi_2(r)\frac{d}{dt}f_2(t)\right],\\F^*(r)&=F(r)-\left[\varphi_1(r)\,f_1(0)+\varphi_2(r)\,f_2(0)\right],\end{aligned} \tag{59}$$

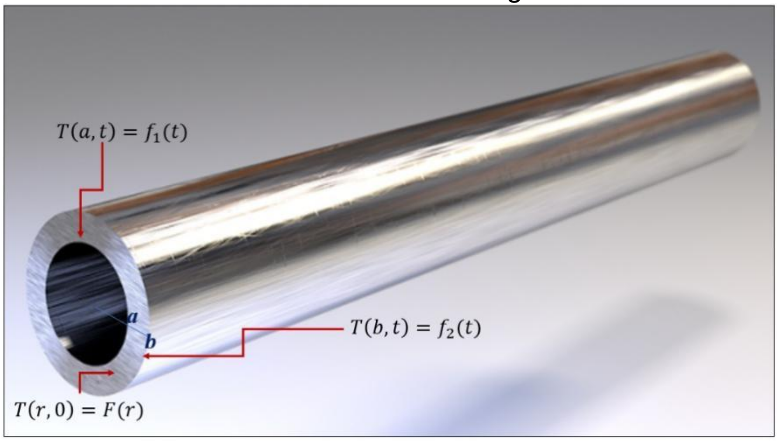
while the λ_n are given by equation (29) as

$$G_0(\lambda r)=\left[J_0(\lambda r)Y_0(\lambda b)-J_0(\lambda b)Y_0(\lambda r)\right]=0.$$

Some special cases of equation (58) will be analyzed using this general equation for an approximation of the solution of problems given by equations (1) to (4).

The Figure 1 shows a long and thin hollow cylinder of homogeneous and isotropic aluminum $L: a \le r \le b$, with Dirichlet conditions dependent only on time and heat generation.

Figure 1 – Long, thin hollow aluminum cylinder with time-dependent Dirichlet conditions on each surface of the solid with heat generation.



Source: Authors' elaboration (2020).

Let's roughly consider the following parameters for aluminum: $k = 190\left(W/m\,^{o}K\right)$, $a = 0.045\ mm,\ b = 0.075\ mm$, and being $\kappa = 80\times10^{-6}\left(m^2/s\right)$. The temperature and time units in International System of Units (SI) were converted to degrees and minutes, respectively.

3.1. Case I.A

Let's consider the cylinder in Figure 1, subject to the following Dirichlet conditions, without heat generation and constant initial condition, *i.e.*,

$$\begin{aligned} &T(a,t) = T(b,t) = 0,\\ &g(r,t) \equiv 0,\\ &F(r) = V. \end{aligned} \tag{60}$$

From equations (59) and (60) we have,

$$F^*(r) = F(r) = V \quad \text{and} \quad g^* \equiv 0. \tag{61}$$

From equations (58), (60) and (61) results the solution of this problem in the form

$$T(r,t) = \frac{V\pi^2}{2}\sum_{n=1}^{\infty}\frac{\lambda_n^2 J_0(\lambda_n a)}{J_0^2(\lambda_n a) - J_0^2(\lambda_n b)} e^{-\kappa\lambda_n^2 t} G_0(\lambda_n r)\int_{r'=a}^{b} r' G_0(\lambda_n r')\,dr'. \tag{62}$$

From equations (46) and (62) we have

$$T(r,t) = V\pi\sum_{n=1}^{\infty} e^{-\kappa\lambda_n^2 t}\frac{J_0(\lambda_n a)}{J_0^2(\lambda_n a) - J_0^2(\lambda_n b)} G_0(\lambda_n r). \tag{63}$$

Equation (63) is in accordance with equation (13) obtained by Carslaw and Jaeger (2011, p. 207) for this case.

3.2. Case I.B

The cylinder in Figure 1 is under the same conditions as Case I.A, but now with uniform heat generation, that is,

$$\begin{aligned} &T(r=a,t)=T(r=b,t)=0,\\ &g(r,t)\equiv g_0,\\ &F(r)=V, \end{aligned} \tag{64}$$

where g_0 is now taken as constant. From the equations (59, 64) we have

$$F^*(r)=F(r)=V \quad \text{and} \quad g^*\equiv \frac{g_0}{k}. \tag{65}$$

From equations (58), (64) and (65) results the solution of this problem in the form

$$\begin{aligned} T(r,t)=&V\pi\sum_{n=1}^{\infty}e^{-\kappa\lambda_n^2 t}\frac{J_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}G_0(\lambda_n r)+\\ &+\frac{\pi^2}{2}\frac{\kappa}{k}g_0\sum_{n=1}^{\infty}e^{-\kappa\lambda_n^2 t}\frac{\lambda_n^2 J_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}G_0(\lambda_n r)\int_{\tau=0}^{t}\int_{r'=a}^{b}e^{-\kappa\lambda_n^2\tau}G_0(\lambda_n r')r'dr'd\tau. \end{aligned} \tag{66}$$

From equations (46) and (66) we obtain the approximate solution of this problem in the form

$$\begin{aligned} T(r,t)=&V\pi\sum_{n=1}^{\infty}e^{-\kappa\lambda_n^2 t}\frac{J_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}G_0(\lambda_n r)+\\ &+\frac{\pi g_0}{k}\sum_{n=1}^{\infty}\left\{\frac{J_0(\lambda_n a)}{\lambda_n^2\left[J_0(\lambda_n a)+J_0(\lambda_n b)\right]}G_0(\lambda_n r)-\frac{J_0(\lambda_n a)}{\lambda_n^2\left[J_0(\lambda_n a)+J_0(\lambda_n b)\right]}G_0(\lambda_n r)e^{-\kappa\lambda_n^2 t}\right\} \end{aligned} \tag{67}$$

Note that equation (67) “carries” the solution given by equation (63) which contained only the contribution of the initial condition. When $t\to\infty$, $T(r,t)\to T_{SS}(r)$, that is, the temperature tends to the steady state $T_{SS}(r)$. The stationary equation in a 1-D hollow cylinder with heat generation can be obtained from equation (15) of Carslaw and Jaeger (2011, p. 191) in the form $T_{SS}(r)=A+B\ln r-\frac{g_0 r^2}{4k}$. Then, and using the boundary conditions of this problem, we can identify the sum of the first term between keys of equation (67) with the stationary solution, that is,

$$\begin{aligned} \frac{\pi g_0}{k}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n a)G_0(\lambda_n r)}{\lambda_n^2\left[J_0(\lambda_n a)+J_0(\lambda_n b)\right]}\equiv&-\frac{g_0}{\ln\left(\frac{a}{b}\right)}\frac{\left(a^2-b^2\right)}{4k}\ln(a)+\frac{g_0 a^2}{4k}+\\ &+\frac{g_0}{\ln\left(\frac{a}{b}\right)}\frac{\left(a^2-b^2\right)}{4k}\ln(r)-\frac{g_0 r^2}{4k}=T_{SS}(r). \end{aligned} \tag{68}$$

Introducing equation (68) into equation (67) we obtain the solution to this problem given by

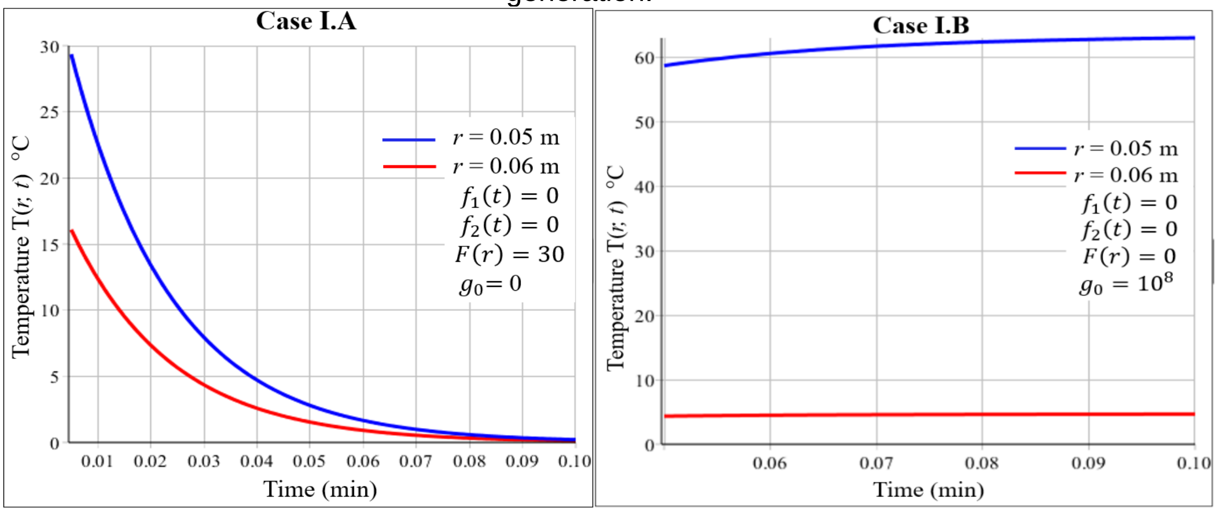
$$T(r,t)=V\pi\sum_{n=1}^{\infty}e^{-\kappa\lambda_n^2 t}\frac{J_0(\lambda_n a)}{J_0(\lambda_n a)+J_0(\lambda_n b)}G_0(\lambda_n r)-\frac{\pi g_0}{k}\sum_{n=1}^{\infty}e^{-\kappa\lambda_n^2 t}\frac{J_0(\lambda_n a)}{\lambda_n^2\left[J_0(\lambda_n a)+J_0(\lambda_n b)\right]}G_0(\lambda_n r)+$$
$$-\frac{g_0}{\ln\left(\frac{a}{b}\right)}\frac{(a^2-b^2)}{4k}\ln(a)+\frac{g_0 a^2}{4k}+\frac{g_0}{\ln\left(\frac{a}{b}\right)}\frac{(a^2-b^2)}{4k}\ln(r)-\frac{g_0 r^2}{4k}. \tag{69}$$

In particular, if $V=0$, the equation (69) would be in the form

$$T(r,t)=-\frac{\pi g_0}{k}\sum_{n=1}^{\infty}e^{-\kappa\lambda_n^2 t}\frac{J_0(\lambda_n a)}{\lambda_n^2\left[J_0(\lambda_n a)+J_0(\lambda_n b)\right]}G_0(\lambda_n r)-\frac{g_0}{\ln\left(\frac{a}{b}\right)}\frac{(a^2-b^2)}{4k}\ln(a)+$$
$$+\frac{g_0 a^2}{4k}+\frac{g_0}{\ln\left(\frac{a}{b}\right)}\frac{(a^2-b^2)}{4k}\ln(r)-\frac{g_0 r^2}{4k}. \tag{70}$$

The Figure 2 shows a simulation with cases I.A and I.B, using equations (63) and (70), respectively.

Figure 2 – Temperature profiles in cylinder 1-D: (a) without heat generation and (b) with heat generation.



Source: Authors' elaboration (2020).

In the case I.A, the more accentuated temperature decay to zero can be noticed near the region in $r=a$ that is under the effect of the null temperature condition on that surface. In case I.B, only the effect of the heat generation term on the temperature profile is noted. Note that graphs were generated close to the boundary in $r=a$ and with small times.

3.3. Case II.A

The cylinder in Figure 1 is subject to the following constant and not necessarily null Dirichlet conditions, without heat generation, and initial condition as a function of the spatial variable, that is,

$$
\begin{aligned}
&T(a,t)=V_1, \quad T(b,t)=V_2,\\
&g(r,t)\equiv 0, \\
&T(r,0)=F(r).
\end{aligned}
\tag{71}
$$

From equations (59) and (71) we have

$$
F^*(r)=F(r)-\left[\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\frac{b}{a}}\right]
\tag{72}
$$

and

$$
g^*(r,t)\equiv 0\,.
\tag{73}
$$

Here, the stationary solution to this heat conduction problem is given by

$$
T_{SS}(r)=\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\frac{b}{a}}\,.
\tag{74}
$$

From equations (58), (72) and (73) we have

$$
\begin{aligned}
T(r,t)=&\frac{\pi^2}{2}\sum_{n=1}^{\infty}\lambda_n^2\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)\int_a^b G_0(\lambda_n r')F(r')r'dr'+\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\frac{b}{a}}+\\
&-\frac{\pi^2}{2}\sum_{n=1}^{\infty}\lambda_n^2\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)\int_a^b G_0(\lambda_n r')\left\{\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\frac{b}{a}}\right\}r'dr'.
\end{aligned}
\tag{75}
$$

Using the integral reported by Carslaw and Jaeger (2011, p. 207) one can obtain

$$
\int_a^b G_0(\lambda_n r')\left(\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\frac{b}{a}}\right)r'dr'=\frac{2\left(-V_1J_0(\lambda_n b)+V_2J_0(\lambda_n a)\right)}{\pi\lambda_n^2J_0(\lambda_n a)}\,.
\tag{76}
$$

From equations (75) and (76) results the expression given by

$$
\begin{aligned}
T(r,t)=&\frac{\pi^2}{2}\sum_{n=1}^{\infty}\lambda_n^2\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)\int_a^b G_0(\lambda_n r')F(r')r'dr'+\\
&-\pi\sum_{n=1}^{\infty}\frac{\left[-V_1J_0(\lambda_n b)+V_2J_0(\lambda_n a)\right]}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}J_0(\lambda_n a)e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)+\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\frac{b}{a}}.
\end{aligned}
\tag{77}
$$

The equation (77) agrees with equation (15) of Carslaw and Jaeger (2011, p. 207) for this case, where the authors used the method of separating variables through a translation of the stationary solution. In particular if in equation (71), $V_1=V_2$ and $F(r)=V_0$, using equation (46), equation (77) can be put in the form

$$T\left(r,t\right)=\pi V_0\sum_{n=1}^{\infty}\frac{J_0\left(\lambda_n a\right)}{J_0\left(\lambda_n a\right)+J_0\left(\lambda_n b\right)}G_0\left(\lambda_n r\right)e^{-\kappa\lambda_n^2 t}-\pi V_1\sum_{n=1}^{\infty}\frac{J_0\left(\lambda_n a\right)}{J_0\left(\lambda_n a\right)+J_0\left(\lambda_n b\right)}G_0\left(\lambda_n r\right)e^{-\kappa\lambda_n^2 t}+V_1. \quad (78)$$

This equation can be easily adapted for application in substance diffusion (CRANK, 2011) replacing κ by D and V_i by the substance concentration C_i.

3.4. Case II.B

The cylinder in Figure 1 is subject to the following constant and not necessarily null Dirichlet conditions, with uniform heat generation, g_0, and initial condition as a function of the spatial variable, *i.e.*,

$$\begin{aligned}&T\left(a,t\right)=V_1, \quad T\left(b,t\right)=V_2,\\ &g\left(r,t\right)\equiv g_0,\\ &T\left(r,0\right)=F\left(r\right).\end{aligned} \quad (79)$$

From equations (59) and (79) we have

$$g*\left(r,t\right)=\frac{g_0}{k}. \quad (80)$$

Introducing the second term, that appears on the right side of equation (58), in equation (77), we get

$$\begin{aligned}T\left(r,t\right)=&\frac{\pi^2}{2}\sum_{n=1}^{\infty}\lambda_n^2\frac{J_0^2\left(\lambda_n a\right)}{J_0^2\left(\lambda_n a\right)-J_0^2\left(\lambda_n b\right)}e^{-\kappa\lambda_n^2 t}G_0\left(\lambda_n r\right)\int_a^b G_0\left(\lambda_n r'\right)F\left(r'\right)r'dr'+\\ &-\pi\sum_{n=1}^{\infty}\frac{\left\{-V_1 J_0\left(\lambda_n b\right)+V_2 J_0\left(\lambda_n a\right)\right\}}{J_0^2\left(\lambda_n a\right)-J_0^2\left(\lambda_n b\right)}J_0\left(\lambda_n a\right)e^{-\kappa\lambda_n^2 t}G_0\left(\lambda_n r\right)+\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\frac{b}{a}}+\\ &+\frac{\kappa\pi^2}{2}\sum_{n=1}^{\infty}\frac{\lambda_n^2 J_0\left(\lambda_n a\right)}{J_0^2\left(\lambda_n a\right)-J_0^2\left(\lambda_n b\right)}e^{-\kappa\lambda_n^2 t}G_0\left(\lambda_n r\right)\int_0^t\int_a^b e^{-\kappa\lambda_n^2\tau}G_0\left(\lambda_n r'\right)g*(r',\tau)r'dr'd\tau.\end{aligned} \quad (81)$$

The double integral that appears in equation (81) can be solved separately into the variable τ and then into the spatial variable r, using equation (46) to obtain

$$\begin{aligned}T\left(r,t\right)=&\frac{\pi^2}{2}\sum_{n=1}^{\infty}\lambda_n^2\frac{J_0^2\left(\lambda_n a\right)}{J_0^2\left(\lambda_n a\right)-J_0^2\left(\lambda_n b\right)}e^{-\kappa\lambda_n^2 t}G_0\left(\lambda_n r\right)\int_a^b G_0\left(\lambda_n r'\right)F\left(r'\right)r'dr'+\\ &-\pi\sum_{n=1}^{\infty}\frac{\left\{-V_1 J_0\left(\lambda_n b\right)+V_2 J_0\left(\lambda_n a\right)\right\}}{J_0^2\left(\lambda_n a\right)-J_0^2\left(\lambda_n b\right)}J_0\left(\lambda_n a\right)e^{-\kappa\lambda_n^2 t}G_0\left(\lambda_n r\right)+\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\frac{b}{a}}+\\ &\left(-\frac{g_0\pi}{k}\sum_{n=1}^{\infty}\frac{J_0\left(\lambda_n a\right)G_0\left(\lambda_n r\right)}{\lambda_n^2\left[J_0\left(\lambda_n a\right)-J_0\left(\lambda_n b\right)\right]}e^{-\kappa\lambda_n^2 t}+\frac{g_0\pi}{k}\sum_{n=1}^{\infty}\frac{J_0\left(\lambda_n a\right)G_0\left(\lambda_n r\right)}{\lambda_n^2\left[J_0\left(\lambda_n a\right)-J_0\left(\lambda_n b\right)\right]}\right).\end{aligned} \quad (82)$$

When $t \to \infty$, $T(r,t) \to T_{SS}(r)$, that is, from equations (68) and (82) the temperature tends to

$$\frac{\pi g_0}{k}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n a)G_0(\lambda_n r)}{\lambda_n^2\left[J_0(\lambda_n a)+J_0(\lambda_n b)\right]}+\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\left(\frac{a}{b}\right)} \equiv -\frac{g_0}{\ln\left(\frac{a}{b}\right)}\frac{(a^2-b^2)}{4k}\ln(a)+\frac{g_0\, a^2}{4k}+$$
$$+\frac{g_0}{\ln\left(\frac{a}{b}\right)}\frac{(a^2-b^2)}{4k}\ln(r)-\frac{g_0\, r^2}{4k}+\frac{V_1\ln\frac{b}{r}+V_2\ln\frac{r}{a}}{\ln\left(\frac{a}{b}\right)}=T_{SS}(r). \tag{83}$$

Equation (83) can still be placed as

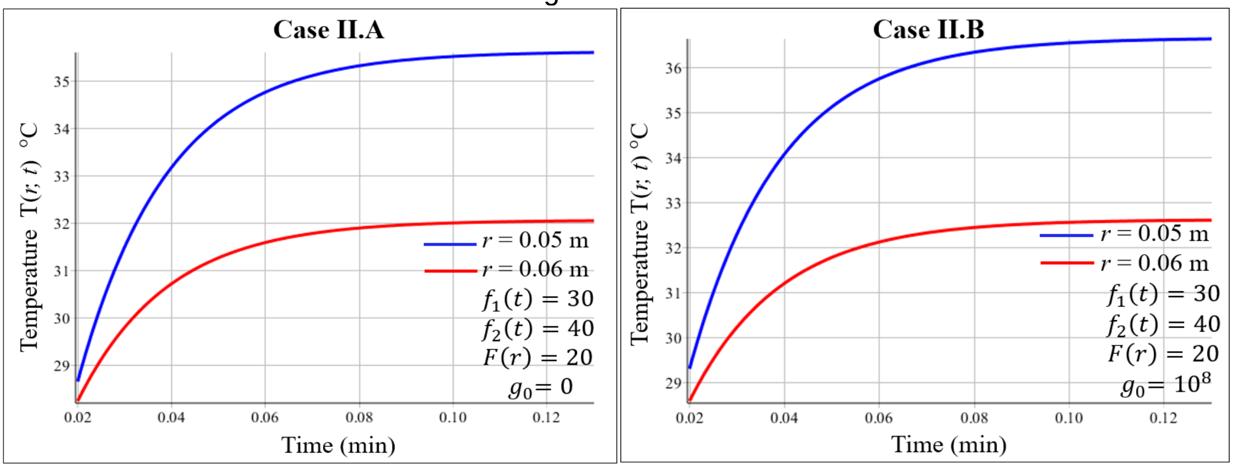
$$T_{SS}(r)=V_1+g_0\frac{(a^2-r^2)}{4k}+\frac{1}{\ln\left(\frac{a}{b}\right)}\left\{V_1-V_2+g_0\frac{(a^2-r^2)}{4k}\right\}\ln\left(\frac{r}{a}\right). \tag{84}$$

It's easy to verify that $T_{SS}(a)=V_1$ and $T_{SS}(b)=V_2$. From equations (82) and (84) we have

$$T(r,t)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\lambda_n^2\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)\int_a^b G_0(\lambda_n r')F(r')r'dr'+$$
$$-\pi\sum_{n=1}^{\infty}\frac{-V_1J_0(\lambda_n b)+V_2J_0(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}J_0(\lambda_n a)e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)+V_1+g_0\frac{(a^2-r^2)}{4k} \quad . \tag{85}$$
$$+\frac{1}{\ln\frac{a}{b}}\left\{V_1-V_2+g_0\frac{(a^2-r^2)}{4k}\right\}\ln\frac{r}{a}-\frac{g_0\pi}{k}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n a)G_0(\lambda_n r)}{\lambda_n^2\left[J_0(\lambda_n a)-J_0(\lambda_n b)\right]}e^{-\kappa\lambda_n^2 t}.$$

Equations (82) and (85) can be used to approximate the solution of the problem given by case II.B. Figure 3 shows a simulation with cases II. A and II. B, using equation (77) and equation (85), respectively.

Figure 3 – Temperature profiles in the cylinder 1-D: (a) without heat generation and (b) with heat generation.

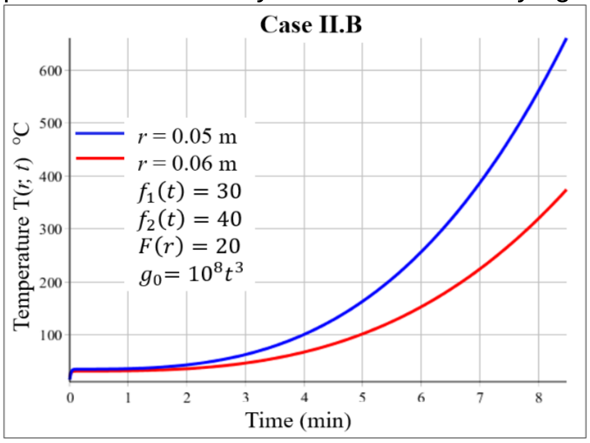


Source: Authors' elaboration (2020).

It can be noted that the uniform heat source did not produce a significant temperature rise in the cylinder, showing that the constant temperature Dirichlet conditions appear to limit the temperature rise produced by the uniform heat source.

Figure 4 shows a simulation of this problem, but now considering a variable source term. To generate this graph, equation (81) was used, since the integration in time does not provide a closed formula as occurred with the uniform heat source in the previous case given by equation (84).

Figure 4 – Temperature profiles in the 1-D cylinder with time-varying uniform heat generation.



Source: Authors' elaboration (2020).

These temperature profiles show a rapid temperature rise for aluminum smelting that occurs around $660\ ^0C$. A greater temperature variation can be noted in the $r = 0.06\,m$ position in relation to the region closest to the inner surface at $r = 0.06\,m,$ in relation to the region closest to the inner surface at $r = a$. This may be due to the greater influence of the Dirichlet condition on the outer surface in $r = b$ which has a higher prescribed temperature.

The last case considers a time-varying Dirichlet condition on the surface in $r = b$.

3.5. Case III

The cylinder in Figure 1 is subject to the following Dirichlet conditions, one being constant and the other variable, with uniform heat generation, g_0, and initial condition of constant value as a function of the spatial variable r, that is,

$$\begin{array}{l} T(a,t) = V_1, \quad T(b,t) = \beta t, \\ g(r,t) \equiv g_0, \\ T(r,0) = V_0, \end{array} \quad . \tag{86}$$

where β is a positive constant. From equations (59) and (86) we have

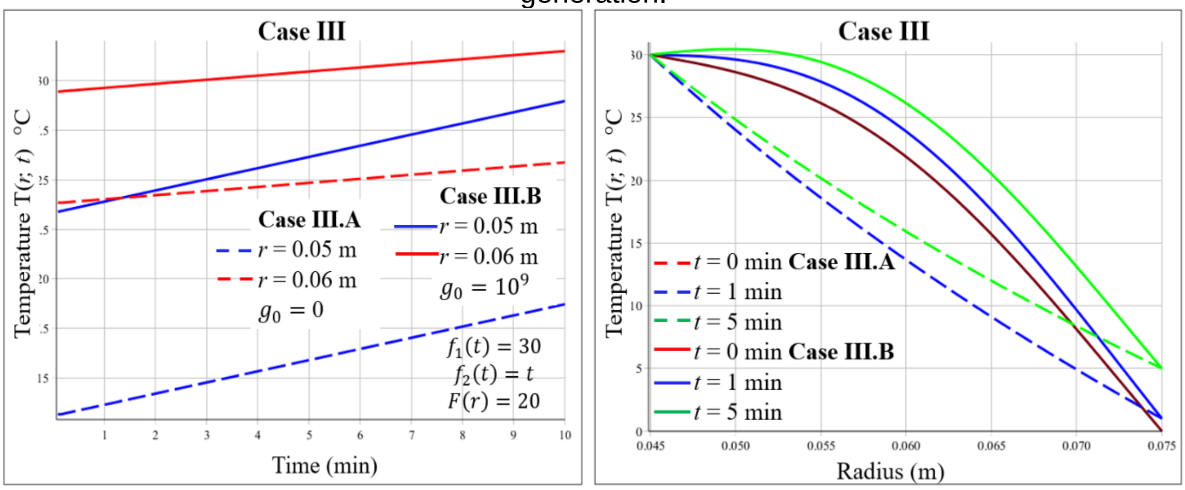
$$g^*(r,t)=\frac{g_0}{k}-\frac{\beta}{\kappa}\left[\frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)}\right] \quad \text{and} \quad F^*(r,t)=V_0-V_1\left[\frac{\ln\left(\frac{b}{r}\right)}{\ln\left(\frac{b}{a}\right)}\right]. \tag{87}$$

From equations (58) and (87) results the expression of the approximate solution of this problem given by

$$\begin{aligned} T(r,t)&=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{\lambda_n^2 J_0^2(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)\int_a^b G_0(\lambda_n r')\left[V_0-V_1\frac{\ln\left(\frac{b}{r}\right)}{\ln\left(\frac{b}{a}\right)}\right]r'dr'+\\ &+\frac{\pi^2}{2}\kappa\sum_{n=1}^{\infty}\frac{\lambda_n^2 J_0^2(\lambda_n a)}{J_0^2(\lambda_n a)-J_0^2(\lambda_n b)}e^{-\kappa\lambda_n^2 t}G_0(\lambda_n r)\int_0^t\int_a^b e^{\kappa\lambda_n^2\tau}G_0(\lambda_n r')\left[\frac{g_0}{k}-\frac{\beta}{\kappa}\frac{\ln\left(\frac{b}{r}\right)}{\ln\left(\frac{b}{a}\right)}\right]r'dr'd\tau\cdot \\ &+V_1\frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)}+\beta\left[\frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)}\right]t. \end{aligned} \tag{88}$$

The Figure 5 shows temperature profiles for case III.A when $g_0 \equiv 0$, compared to case III.B with g_0 constant. The graphs were generated with equation (88). Figure 5(a) shows the temperature distribution for cases III.A and III.B in two cylinder positions, while Figure 5(b) shows some temperature profiles along the spatial domain with fixed time. In this figure it can be noticed that on the surfaces of the hollow cylinder, the temperatures converge to the same value as the Dirichlet conditions, with or without heat. This fact was expected and shows the quality of the solution obtained for this problem with equation (88).

Figure 5 – Temperature profiles in the 1-D cylinder (a) without heat generation and (b) with heat generation.



Source: Authors' elaboration (2020).

4. Conclusions

An approximately general solution for the transient 1-D heat conduction problem, with heat generation in homogeneous and isotropic hollow cylinders, with Dirichlet condition as a function of time on the surfaces, and a given initial condition was established here. To alleviate or mitigate possible difficulties in the convergence of the solution near the surfaces that constitute the temperature distribution region, a known method was adopted which consists of transforming the non-homogeneous boundary conditions into homogeneous ones, which combined with the obtained Green's function from the approximate solution of the homogeneous version of the analyzed problem, it became possible to obtain an approximately general solution for this type of problem. Simulations were performed with this equation, some of them involving special cases reported in the literature, which allowed the comparison of the obtained results, observing a good agreement between them. Thus, this solution allows the analysis of more general thermal problems that may have applications in engineering. Furthermore, this solution can be used to validate approximate numerical solutions, increasing their accuracy. An important integral formed by Bessel functions integrating the general expression was solved analytically, and its result used in the general solution to obtain simplified expressions of the special cases analyzed here.

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