Gaussian integral by Taylor series and applications

Integral gaussiana pela série de Taylor e aplicações

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Abstract
In this paper, we present a solution for a specific Gaussian integral. Introducing a parameter that depends on a n index, we found out a general solution inspired by the Taylor series of a simple function. We demonstrated that this parameter represents the expansion coefficients of this function, a very interesting and new result. We also introduced some Theorems that are proved by mathematical induction. As a test for the solution presented here, we investigated a non-extensive version for the particle number density in Tsallis framework, which enabled us to evaluate the functionality of the method. Besides, solutions for a certain class of the gamma and factorial functions are derived. Moreover, we presented a simple application in fractional calculus. In conclusion, we believe in the relevance of this work because it presents a solution for the Gaussian integral from an unprecedented perspective.

Palavras-chave
Integral Gaussiana
Funções Especiais
Derivada Fracionária

Resumo
Neste artigo, apresentamos uma solução para uma integral gaussiana específica. Introduzindo um parâmetro que depende de um índice n, encontramos uma solução geral inspirada na série Taylor de uma função simples. Demonstramos que esse parâmetro representa os coeficientes da expansão dessa função, um resultado muito interessante e novo. Também introduzimos alguns teoremas que são provados por indução matemática. Como teste para a solução apresentada aqui, investigamos uma versão não extensiva para a densidade do número de partículas na estrutura de Tsallis, o que nos permitiu avaliar a funcionalidade do método. Soluções para uma determinada classe das funções gama e fatorial também são derivadas. Além disso, apresentamos uma aplicação simples em cálculo fracionário. Concluindo, acreditamos na relevância deste trabalho, pois apresenta uma solução para a integral gaussiana de uma perspectiva inédita.

1 Introduction

Mathematics is present in the teaching and research of several other areas such as physics and engineering. Mathematical knowledge is necessary to solve problems that can understand some phenomena. One of the areas of application is the theoretical physics that has been developed and following the advance of research in mathematics, especially in the fields of geometry and analysis. In this sense, we can speak of mathematical physics as an area of physical interest that involves mathematical knowledge. This area comprises contents such as tensors, mathematical analysis, field theory, among others. As a highlight, we mention the Gaussian integral that also is known as probability integral being the integral of the function $\exp(-x^2)$ over the entire line $(-\infty,\infty)$. Solutions of this type of integral involve the so-called gamma functions introduced by Euler in the 18th century and improved by Legendre, Gauss, and Weierstrass (DAVIS, 1959, GRONAU, 2003). The Gaussian integral has a wide range of applications in several areas of knowledge. Indeed, when we do a slight change of variables it is possible to compute the normalizing constant of the normal distribution in probability and statistics (SPIEGEL; SCHILLER; SRINIVASAN, 2001, STAHL, 2006). In physics the Gaussian integral appears frequently in quantum mechanics (GREINER, 1990), to find the probability density of the ground state of the harmonic oscillator, in the path integral formulation (SAKURAI, 1985), to find the propagator of the harmonic oscillator and in statistical mechanics (GREINER, 1995, PATHRIA, 1996, SALINAS, 2001), to find its partition function.

2 Theoretical reference

Consider the solution of the following Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}\,, \tag{1}$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$. There are several methods to solve this type of integral, but the most well known and widely used in textbooks is the double integral method (STURM, 1857). Other methods can be found in Laplace (1820) and Stigler (1986). Conrad (2013) synthesized eleven ways for solving the Gaussian integral, among them, one uses the method of Fourier transforms, Stirling's formula, contour integration, and differentiation under the integral sign. Let us now consider a more general Gaussian integral like (SALINAS, 2001, HERNANDEZ, 2015, WEISSTEIN, 2020[a])

$$\int_{-\infty}^{+\infty} x^{2n}\, e^{-\alpha x^2}\, dx = \frac{(2n-1)!!}{2^n}\,\frac{\sqrt{\pi}}{\alpha^{\frac{2n+1}{2}}}\,,\ \ (\forall\, n \in \mathbb{N})\,, \tag{2}$$

or

$$\int_{-\infty}^{+\infty} x^{2n}\, e^{-\alpha x^2}\, dx\ = \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\alpha^{\frac{2n+1}{2}}}\,,\ \ (\forall\, n \in \mathbb{N})\,. \tag{3}$$

For reasons of simplicity, we are considering that the number zero is included in the set of natural numbers. Here, $m!!$ is the double factorial that is defined as follow (ARFKEN; WEBER, 2005):

$$m!! = \begin{cases} m\cdot(m-2)\ldots 5\cdot 3\cdot 1 & m>0 \text{ odd} \\ m\cdot(m-2)\ldots 6\cdot 4\cdot 2 & m>0 \text{ even} \\ 1 & m=-1,\ 0\ , \end{cases} \tag{4}$$

and $\Gamma(x)$ is the gamma function given by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt\ . \tag{5}$$

The gamma function has great relevance for the development of new functions that can be applied directly to physics. Normally this function is present in problems of physics such as, for example, in the normalization of Coulomb wave functions and the calculation of probabilities in statistical mechanics (ARFKEN; WEBER, 2005). Notice that formulas (2) and (3) admits a solution for x^{2n+1} since x ranges from zero to infinity. However, we present only the solutions for x^{2n} because the solution which will be presented in this work requires it.

3 Methodology

Inspired by the ways to solve the Gaussian integral as presented by Conrad, this work aims at presenting a solution for a specific Gaussian integral based on the Taylor Series of a simple function. This is relevant because it enables us to treat certain mathematical and physical problems from a new perspective having the differential calculus as a tool. The proposed solution consists of introducing a parameter that has the role of describing the evolution of the numerical sequence of Gaussian integral. In other words, the coefficient that accompanies $\sqrt{\pi}$ (for example with $\alpha = 1$) gives us this numerical sequence. Through an analysis of the expansion in a Taylor series of the function $f(x) = (1-x)^{-1/2}$, we will show that this parameter can be thought of as the coefficients of this expansion. Having done this, we were able to introduce some Theorems that are proved by mathematical induction. As a test for the method, we obtained a non-extensive version for the particle number density via Tsallis statistics. Also, solutions for the gamma function of the form $\Gamma(1/2\pm n)$, and the factorial function of the type $(1/2\pm n)!$ was acquired, where $n\in\mathbb{N}$, are derived. Besides, we present an application in fractional calculus using the definitions of fractional derivative according to Riemann-Liouville and Caputo.

4 Results and discussions

Consider the Gaussian integral as follows:

$$I_{2n} = \int_{-\infty}^{+\infty} x^{2n}\, e^{-\alpha x^2}\, dx\,, \quad (\forall\, n \in \mathbb{N}; \alpha > 0)\,. \tag{6}$$

Solving I_{2n} by conventional methods, we achieved the following results: $I_0 = \sqrt{\pi}/\alpha^{1/2}$, $I_2 = \sqrt{\pi}/2\alpha^{3/2}$, $I_4 = 3\sqrt{\pi}/4\alpha^{5/2}$, $I_6 = 15\sqrt{\pi}/8\alpha^{7/2}$ and so on. We can generalize these results by introducing a parameter that depends on n. The solution proposed here is based on the following definition:

$$I_{2n} \equiv \gamma_{2n} \frac{\sqrt{\pi}}{\alpha^{\frac{2n+1}{2}}}\,, \tag{7}$$

where γ_{2n} is a parameter to be determined. Note that $\gamma_0 = 1$, $\gamma_2 = 1/2$, $\gamma_4 = 3/4$, $\gamma_6 = 15/8$ and so forth. The objective of this research is to find an expression for sequence, but it becomes evident when we perform the expansion of the function $f(x) = (1-x)^{-1/2}$ in a Taylor series around $x = 0$, as below:

$$f(x) = \sum_{n=0}^{\infty} \left.\frac{d^n(1-x)^{-1/2}}{dx^n}\right|_{x=0} \frac{x^n}{n!}. \tag{8}$$

Analyzing the first coefficients of the expansion above, the following results were achieved: $f(0) = 1$, $f'(0) = 1/2$, $f''(0) = 3/4$, $f'''(0) = 15/8$, $f''''(0) = 105/16$ and so on. A meaningful remark is that the coefficients of the expansion, that is $f^n(0)$, generate identical numbers which were generated by the by the evolution of the parameter γ_{2n}, as seen previously. Thus, it is reasonable to establish the following definition:

Definition 1. For $n \in \mathbb{N}$, γ_{2n} can be defined as

$$\gamma_{2n} \equiv \left.\frac{d^n(1-x)^{-1/2}}{dx^n}\right|_{x=0}. \tag{9}$$

In the next Proposition we prove a formula that can be used to calculate the derivative of any order for the function $f(x) = (1-x)^{-1/2}$, that appears in the Definition 1.

Proposition 1. *The following statement is valid for all* $n \in \mathbb{N}$.

$$\frac{d^n(1-x)^{-1/2}}{dx^n} = \left[\prod_{k=0}^{n-1}\left(\frac{1}{2}+k\right)\right].(1-x)^{-\frac{1}{2}-n}.$$

Proof. The demonstration will be done by induction over n. For $n = 1$, we have, by the chain rule that:

$$\frac{d(1-x)^{-1/2}}{dx} = -\frac{1}{2}.(1-x)^{-\frac{1}{2}-1}.(-1) = \frac{1}{2}.(1-x)^{-\frac{1}{2}-1}.$$

Now, suppose that for any natural number $n < s+1 \in \mathbb{N}$ we have

$$\frac{d^n(1-x)^{-1/2}}{dx^n} = \left[\prod_{k=0}^{n-1}\left(\frac{1}{2}+k\right)\right].(1-x)^{-\frac{1}{2}-n}.$$

We will now show that the equation above is also valid for $n = s+1$. Indeed, deriving $\frac{d^s(1-x)^{-1/2}}{dx^s}$ we obtain:

$$\begin{aligned}\frac{d^{s+1}(1-x)^{-1/2}}{dx^{s+1}} &= \frac{d\left\{\left[\prod_{k=0}^{s-1}\left(\frac{1}{2}+k\right)\right].(1-x)^{-\frac{1}{2}-s}\right\}}{dx} \\ &= \left[\prod_{k=0}^{s-1}\left(\frac{1}{2}+k\right)\right].\left(\frac{1}{2}+s\right).(1-x)^{-\frac{1}{2}-s-1} \\ &= \left[\prod_{k=0}^{s}\left(\frac{1}{2}+k\right)\right].(1-x)^{-\frac{1}{2}-(s+1)}.\end{aligned}$$

This ends this proof. □

Now, we will use Proposition 1 to demonstrate the following Lemma:

Lemma 1. *If γ_{2n} is the function of Definition 1, then for all $n \in \mathbb{N}$ we have*

$$\gamma_{2n} = \prod_{k=0}^{n-1}\left(\frac{1}{2}+k\right) . \tag{10}$$

Proof. Straightforward from Definition 1 and Proposition 1.

□

Definition 2. From (2) we can write

$$\bar{\gamma}_{2n} \equiv \frac{(2n-1)!!}{2^n} , \ \forall \ n \in \mathbb{N} . \tag{11}$$

Theorem 1. *Let I_{2n} given by (2), then*

$$I_{2n} = \gamma_{2n} \frac{\sqrt{\pi}}{\alpha^{\frac{2n+1}{2}}} , \ \forall \ n \in \mathbb{N} , \tag{12}$$

where γ_{2n} is given by Definition 1.

To prove the Theorem 1, we will take into account the Definition 2. In other words, we just need to show that $\gamma_{2n} = \bar{\gamma}_{2n}$, for all $n \in \mathbb{N}$.

Proof. The base case ($n = 0$) shows that $\gamma_0 = \bar{\gamma}_0$, since from Definition 2, $(-1)!! = 1$. Suppose that $\gamma_{2n} = \bar{\gamma}_{2n}$ is true for $n = q$ with $q \in \mathbb{N}$, then the inductive hypothesis is given by

$$\gamma_{2q} = \bar{\gamma}_{2q} . \tag{13}$$

We now must show that $\gamma_{2n} = \bar{\gamma}_{2n}$ it is also true for $n = q+1$. So we have

$$\gamma_{2(q+1)} = \bar{\gamma}_{2(q+1)} \, . \tag{14}$$

By Lemma 1, with $n = q+1$ and $k = i$, the left-hand side of relation (14) becomes

$$\begin{aligned}
\gamma_{2(q+1)} &= \prod_{i=0}^{q} \left(\frac{1}{2} + i\right) \, , \\
&= \frac{1}{2}\left(\frac{1}{2}+1\right) \cdots \left(\frac{1}{2} + (q-1)\right)\left(\frac{1}{2}+q\right) \, , \\
&= \gamma_{2q}\left(\frac{1}{2}+q\right) \, .
\end{aligned} \tag{15}$$

Applying the inductive hypothesis, we get

$$\gamma_{2(q+1)} = \bar{\gamma}_{2q}\left(\frac{1}{2}+q\right) \, . \tag{16}$$

Using the Definition 2 in (16), we obtain

$$\begin{aligned}
\gamma_{2(q+1)} &= \frac{(2q-1)!!}{2^q}\left(\frac{1}{2}+q\right) \, , \\
&= \frac{1}{2^{q+1}}\left[1 \cdot 3 \cdot 5 \cdots (2q-1) \cdot (2q+1)\right] \, , \\
&= \bar{\gamma}_{2(q+1)} \, .
\end{aligned} \tag{17}$$

This completes the proof.

□

An immediate consequence of the Theorem 1 is that we can establish a relationship between the double factorial and the parameter γ_{2n}, as below:

$$(2n-1)!! = 2^n \left. \frac{d^n (1-x)^{-1/2}}{dx^n} \right|_{x=0} \, , \ \forall \ n \in \mathbb{N} \, . \tag{18}$$

For example, when $n = 1$, we have $1!! = 1$, since $\gamma_2 = 1/2$. For $n = 2$, we have $3!! = 3$, since $\gamma_4 = 3/4$ and so on.

Since there are other methods, this one seems to be more practical than some methods found in the literature. For example, in (3), we need to solve a gamma function $\Gamma(t)$ as the Definition 3 shows, whereas our solution just needs to derive a simple function. Besides, we demonstrated that parameter γ_{2n} represents the Taylor series coefficients of the function $f(x) = (1-x)^{-1/2}$. This means that the evolution of Gaussian integral of a kind I_{2n} has a strong relationship to the expansion terms of $f(x)$, a newsworthy result. One aspect that makes this solution accessible is the fact that, for example, in physical problems, n is not usually very large, which facilitates its application. Hereafter, we briefly present an application in physics, specifically in the determination of the particle number density in the Tsallis framework. Moreover, we present solutions for the gamma and factorial functions (special functions) in terms of parameter γ_{2n}, and we also show a simple application in fractional calculus.

4.1 **Particle number density in Tsallis framework**

We chose to evaluate the particle number density in the Tsallis framework because an ideal scenario emerges in which it is possible to notice the functionality of our solution. In 1988, Constantino Tsallis proposed a possible generalization of the Boltzmann-Gibbs (BG) entropy (TSALLIS, 1988). The proposed new entropy is expressed by

$$S_q = \frac{k_B}{q-1}\left(1-\sum_{i=1}^{\Omega} p_i^q\right) , \tag{19}$$

where k_B is the Boltzmann constant, p_i is the probability of the system to be found in the microstate i and q is the parameter that characterizes the degree of nonextensivity of the system. The classical entropy is recovered in the limit $q \to 1$. In Tsallis' statistics, the particle number of species i per volume can be written as follows:

$$n_i^q = \frac{g_i}{(2\pi\hbar)^3}\int_{-\infty}^{+\infty} d^3p\mathcal{N}_i^q , \tag{20}$$

where $\mathcal{N}_i^q$ is the generalized occupation number, g_i is the degeneracy of species i and $\hbar$ is the Planck reduced constant. The generalized occupation number for fermions in Tsallis framework is given by (SHEN; ZHANG; WANG, 2017)

$$\mathcal{N}_i^q = \frac{1}{e_{2-q}^{\beta(E_i-\mu_i)}+1} , \tag{21}$$

where $e_{2-q}^x \equiv [1+(q-1)x]^{1/(q-1)}$. Further, $\beta = 1/k_BT$, μ_i and E_i are the chemical potential and particle energy of species i, respectively.

Expanding (21) up to the first order of $(q-1)$, we obtain (PESSAH, 2001)

$$\mathcal{N}_i^q = \frac{1}{e^{\beta(E_i-\mu_i)}+1} + \frac{(q-1)}{2}\frac{(\beta(E_i-\mu_i))^2 e^{\beta(E_i-\mu_i)}}{\left(e^{\beta(E_i-\mu_i)}+1\right)^2} . \tag{22}$$

Considering the case in which $k_BT < (E_i-\mu_i)$, then $e^{\beta(E_i-\mu_i)} \gg 1$. Thus, (22) becomes

$$\mathcal{N}_i^q = e^{-\beta(E_i-\mu_i)} + \frac{(q-1)}{2}(\beta(E_i-\mu_i))^2 e^{-\beta(E_i-\mu_i)} . \tag{23}$$

In this way, assuming the energy for non-relativistic particles as $E_i = m_i c^2 + p^2/2m_i$ and using (23) in (20), the generalized particle number density takes the form

$$
\begin{aligned}
n_i^q \quad = \quad & \frac{2\pi g_i}{(2\pi\hbar)^3} e^{\beta(\mu_i - m_i c^2)} \int_{-\infty}^{+\infty} dp\ p^2 e^{-\alpha p^2} \\
& + \frac{2\pi g_i}{(2\pi\hbar)^3} \frac{(q-1)}{2} \beta^2 e^{\beta(\mu_i - m_i c^2)} (m_i c^2 - \mu_i)^2 \int_{-\infty}^{+\infty} dp\ p^2 e^{-\alpha p^2} \\
& + \frac{2\pi g_i}{(2\pi\hbar)^3} \frac{(q-1)}{2} \beta^2 \times m_i (m_i c^2 - \mu_i) e^{\beta(\mu_i - m_i c^2)} \int_{-\infty}^{+\infty} dp\ p^4 e^{-\alpha p^2} \\
& + \frac{g_i}{(2\pi\hbar)^3} \frac{(q-1)}{2} \beta^2 e^{\beta(\mu_i - m_i c^2)} \frac{\pi}{2m_i^2} \int_{-\infty}^{+\infty} dp\ p^6 e^{-\alpha p^2} \ ,
\end{aligned} \tag{24}
$$

where $\alpha \equiv \beta/2m_i$ and c is the speed of light in vacuum.

Note that in the above expression we have four Gaussian integrals of type I_{2n}. It is at this point that we will apply our solution. To solve these Gaussian integrals, we will use the Theorem 1. The results are:

$$\int_{-\infty}^{+\infty} dp\ p^2 e^{-\alpha p^2} = \gamma_2 (2m_i k_B T)^{3/2} \sqrt{\pi}\ , \tag{25}$$

$$\int_{-\infty}^{+\infty} dp\ p^4 e^{-\alpha p^2} = \gamma_4 (2m_i k_B T)^{5/2} \sqrt{\pi}\ , \tag{26}$$

$$\int_{-\infty}^{+\infty} dp\ p^6 e^{-\alpha p^2} = \gamma_6 (2m_i k_B T)^{7/2} \sqrt{\pi}\ . \tag{27}$$

We will use the Definition 1 to compute the coefficients γ_{2n}, thus we obtain $\gamma_2 = 1/2$, $\gamma_4 = 3/4$ and $\gamma_6 = 15/8$. Finally, we can find the generalized particle number density. After a little algebra, we get (PESSAH, 2001)

$$
\begin{aligned}
n_i^q \quad = \quad & g_i \left(\frac{m_i k_B T}{2\pi\hbar^2} \right)^{3/2} e^{\frac{\mu_i - m_i c^2}{k_B T}} \left\{ 1 + \frac{(q-1)}{2} \left[\left(\frac{m_i c^2 - \mu_i}{k_B T} \right)^2 \right. \right. \\
& \left. \left. + \ 3 \left(\frac{m_i c^2 - \mu_i}{k_B T} \right) + \frac{15}{4} \right] \right\} .
\end{aligned}
$$

Notice that the usual particle number density is recovered when $q = 1$.

We chose this application because it arises integrals in which the n index assumes three different values that facilitated the evaluation. As mentioned before, the solution presented here is viable to apply in mathematical and physical problems where the n index is small, since we would need to perform successive derivatives to obtain the result. Even that n is large, it was demonstrated that the solution also works for all $n \in \mathbb{N}$.

4.2 **Special functions**

An immediate application of Theorem 1 is verified in the gamma and factorial functions. The definition of the special functions we will consider from now on is presented below.

Definition 3. Let $t \in \mathbb{R}$, and $t > 0$, the gamma function is defined by (BOAS, 2006, RILEY; HOBSON; BENCE, 2006)

$$\Gamma(t) = \int_0^{+\infty} x^{t-1}\, e^{-x}\, dx\,. \tag{28}$$

Using $t = p+1$, and integrating by parts, we obtain the following recurrence relation:

$$\Gamma(p+1) = p\Gamma(p)\,. \tag{29}$$

Definition 4. For $m > -1$, the factorial function is defined by (BOAS, 2006, RILEY; HOBSON; BENCE, 2006)

$$m! = \int_0^{+\infty} x^m\, e^{-x}\, dx\,. \tag{30}$$

We begin with the gamma function given by Definition 3, setting $t = n + 1/2$, that is

$$\Gamma\left(\frac{1}{2}+n\right) = \int_0^{+\infty} x^{\frac{2n-1}{2}}\, e^{-x}\, dx\,. \tag{31}$$

Taking $x = r^2$, we obtain

$$\Gamma\left(\frac{1}{2}+n\right) = 2\int_0^{+\infty} r^{2n}\, e^{-r^2}\, dr\,.$$

Using the fact that if $f : \mathbb{R} \to \mathbb{R}$ is even and the integrals converge,

$$\int_{-\infty}^{+\infty} f(x)dx\; = 2\int_0^{+\infty} f(x)\, dx\,,$$

we can use the Theorem 1 (with $\alpha = 1$), to get

$$\Gamma\left(\frac{1}{2}+n\right) = \left.\frac{d^n(1-x)^{-1/2}}{dx^n}\right|_{x=0} \sqrt{\pi}\,. \tag{32}$$

Having presented (32), it is convenient to introduce a recurrence relation for the parameter γ_{2n}.

Lemma 2. *If γ_{2n} is given by Definition 1, then*

$$\gamma_{2(n+1)} = \frac{2n+1}{2}\gamma_{2n}\,. \tag{33}$$

Proof. Using (32), and the succeeding term, given by

$$\Gamma\left(\frac{2n+1}{2}+1\right) = \gamma_{2(n+1)}\sqrt{\pi}\,, \tag{34}$$

we can insert these expressions in (29), defining $p = n + 1/2$. Therefore

$$\gamma_{2(n+1)} = \frac{2n+1}{2}\gamma_{2n}\,. \tag{35}$$

□

The Theorem 1 ensures us that the parameter γ_{2n} is given by Definition 1. Based on this, it is straightforward to show that the following equality is valid:

$$\sum_{n=0}^{\infty} \gamma_{2n} \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n x^n \binom{-\frac{1}{2}}{n} , \tag{36}$$

where $\binom{a}{b}$ is the binomial coefficient. From the expression above, we can conclude that

$$\gamma_{2n} = (-1)^n \frac{\sqrt{\pi}}{(-\frac{1}{2}-n)!} , \tag{37}$$

which yields the following result:

$$\Gamma\left(\frac{1}{2}-n\right) = (-1)^n \left(\left. \frac{d^n(1-x)^{-1/2}}{dx^n} \right|_{x=0} \right)^{-1} \sqrt{\pi} , \tag{38}$$

where we use the identity $m! = \Gamma(m+1)$. Therefore, the product of (32) and (38) yields

$$\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}-n\right) = (-1)^n \pi . \tag{39}$$

We now apply Theorem 1 to the factorial function given by Definition 4, putting $m = n+1/2$. Then we have

$$\left(\frac{1}{2}+n\right)! = \int_0^{+\infty} x^{\frac{2n+1}{2}} e^{-x}\, dx . \tag{40}$$

Replacing $x = r^2$ and using Theorem 1 together with Lemma 2, we find

$$\left(\frac{1}{2}+n\right)! = \left(\frac{1}{2}+n\right) \left. \frac{d^n(1-x)^{-1/2}}{dx^n} \right|_{x=0} \sqrt{\pi} . \tag{41}$$

Also,

$$\left(\frac{1}{2}-n\right)! = (-1)^n \left(\frac{1}{2}-n\right) \left(\left. \frac{d^n(1-x)^{-1/2}}{dx^n} \right|_{x=0} \right)^{-1} \sqrt{\pi} . \tag{42}$$

Hence, we have the following product:

$$\left(\frac{1}{2}+n\right)! \left(\frac{1}{2}-n\right)! = (-1)^n \left(\frac{1}{4}-n^2\right) \pi . \tag{43}$$

4.3 Fractional derivative

Here we intend to present an interesting result that may be useful to fractional calculus. Initially let us consider the following definitions:

Definition 5. Let $Re(c) > Re(b) > 0$, the hypergeometric function is defined by (WEISSTEIN, 2020[b])

$${}_2F_1\,(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \tau^{b-1}(1-\tau)^{c-b-1}(1-z\tau)^{-a} d\tau , \tag{44}$$

where $Re(x)$ is real part of x.

Definition 6. The fractional derivative according to Riemann-Liouville is defined by (OLIVEIRA; MACHADO, 2014)

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau \,, \tag{45}$$

where α is a complex number such that $Re(\alpha) > 0$ and $m-1 < Re(\alpha) \leq m$ with $m \in \mathbb{N}$.

Definition 7. The fractional derivative according to Caputo is given by (OLIVEIRA; MACHADO, 2014)

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} f(\tau) d\tau \,, \tag{46}$$

being $Re(\alpha) > 0$ such that $m-1 < Re(\alpha) \leq m$ with $m \in \mathbb{N}$.

By using $f(t) = t^{\theta}$ with $\theta > -1$ in Definition 6, we obtain

$$D^{\alpha} t^{\theta} = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\alpha+1)} t^{\theta-\alpha} \,. \tag{47}$$

For the case where $\alpha = \theta$, we have

$$\frac{d^{\alpha} t^{\alpha}}{dt^{\alpha}} = \alpha! \,. \tag{48}$$

Let us now show that the result above can be represented by the k-th derivative of a simple function applied to a point.

Lemma 3. *Let $k \in \mathbb{N}$, then*

$$\left. \frac{d^k (1-x)^{-1}}{dx^k} \right|_{x=0} = k! \,. \tag{49}$$

Proof. In fact,

$$\begin{aligned} \frac{d^k (1-x)^{-1}}{dx^k} &= \Gamma(k+1) \sum_{n=k}^{\infty} \binom{n}{k} x^{n-k} \,, \\ &= \frac{k!}{(1-x)^{k+1}} \,, \end{aligned} \tag{50}$$

hence, at $x = 0$, we obtain

$$\left. \frac{d^k (1-x)^{-1}}{dx^k} \right|_{x=0} = k! \,. \tag{51}$$

□

Suppose that k can takes the form $k = n + 1/2$, the Lemma 3 may be written as follow:

$$\left. \frac{d^{\frac{2n+1}{2}} (1-x)^{-1}}{dx^{\frac{2n+1}{2}}} \right|_{x=0} = \left(\frac{1}{2} + n \right)! \,. \tag{52}$$

Let us now analyze the left-hand side of the equation above. To accomplish this, we will use the Definitions 6 and 7 to evaluate the fractional derivatives. Thus, using the Definition 6 with $\alpha = n + 1/2$, we find the following result:

$$D^{\frac{2n+1}{2}} (1-x)^{-1} = \frac{x^{-n-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-n\right)} \, {}_2F_1 \left(1, 1; m-n+\frac{1}{2}; x \right) \,, \tag{53}$$

where ${}_2F_1\left(a,b;c;z\right)$ is the hypergeometric function given by the Definition 5. This result applied to the point $x=0$ diverges.

Now using the Definition 7 with $\alpha = n+1/2$ on the left-hand side of (52), we obtain

$$D^{\frac{2n+1}{2}}\left(1-x\right)^{-1} = \frac{\Gamma(m+1)}{\Gamma\left(m-n+\frac{1}{2}\right)}x^{m-n-\frac{1}{2}}\ {}_2F_1\left(m+1,1;m-n+\frac{1}{2};x\right)\,. \tag{54}$$

Note that at the point $x=0$, the result above goes to zero, since $m > n+1/2$. We show that both Riemann-Liouville and Caputo's definitions are flimsy when computed the fractional derivatives of the function $f(x) = (1-x)^{-1}$ at the point $x=0$. On the other hand, this problem can be avoided considering the result presented in (41). Hence, we may write

$$D^{\frac{2n+1}{2}}\left(1-x\right)^{-1}\Big|_{x=0} = \left(\frac{1}{2}+n\right)\left.\frac{d^n(1-x)^{-1/2}}{dx^n}\right|_{x=0}\sqrt{\pi}\,. \tag{55}$$

5 Final considerations

It was presented a solution for solving the Gaussian integral inspired by expansion in Taylor series of a simple function, namely $f(x) = (1-x)^{-1/2}$. Introducing a parameter with a n index dependence, we found a general solution able to be used in any situation, since the Gaussian integral is of type I_{2n}. We demonstrated that the parameter γ_{2n} represents the Taylor series coefficients of the function $f(x)$, a newsworthy result. The reliability of the solution is guaranteed through the proof of some Theorems, which were proved by mathematical induction. To check the functionality of the solution, we obtained the particle number density in the Tsallis framework. Besides, we presented solutions for the gamma function of the form $\Gamma(1/2 \pm n)$, and the factorial function of a kind $(1/2 \pm n)!$, in terms of the parameter γ_{2n}. We also shown that our solution is useful for fractional calculus, for example, the definitions of fractional derivative according to Riemann-Liouville and Caputo are flimsy when evaluated the fractional derivatives of the function $f(x) = (1-x)^{-1}$ at the point $x=0$, whereas according to our solution, we found the result on the functionality and efficiency of that solution, as (55) shows. In conclusion, we believe in the relevance of the present work because it reveals an unprecedented solution for the Gaussian integral: one of the most famous integrals of the exact sciences.

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